ANALYSIS FOR PDEs SET

Volume 2

Continuous Functions

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Selected Excerpts :

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Introduction

Objective. This book is the second of six volumes in a series dedicated to the mathematical tools for solving partial differential equations derived from physics:

Volume 1: *Banach, Fréchet, Hilbert and Neumann Spaces*;

Volume 2: *Continuous Functions*;

Volume 3: *Distributions*;

Volume 4: *Lebesgue and Sobolev Spaces*;

Volume 5: *Traces*;

Volume 6: *Partial Differential Equations.*

This second volume is devoted to the partial differentiation of functions and the construction of primitives, which is its inverse mapping, and to their properties, which will be useful for constructing distributions and studying partial differential equations later.

Target audience. We intended to find simple methods that require a minimal level of knowledge to make these tools accessible to the largest audience possible – PhD candidates, advanced students[1] and engineers – without losing generality and even generalizing some standard results, which may be of interest to some researchers.

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1. **Students?** What might I have answered if one of my MAS students in 1988 had asked for more details about the *de Rham duality theorem* that I used to obtain the pressure in the Navier–Stokes equations? Perhaps I could say that "Jacques-Louis LIONS, my supervisor, wrote that it follows from the de Rham cohomology theorem, of which I understand neither the statement, nor the proof, nor why it implies the result that we are using." What a despicably unscientific appeal to authority!
This question was the starting point of this work: writing proofs that I can explain to my students for every result that I use. It took me 5 years to find the "elementary" proof of the *orthogonality theorem* (Theorem 9.2, p. 194) on the existence of the primitives of a field q. I needed a way to obtain $\oint_\Gamma q \cdot \mathrm{d}\ell = 0$ for every closed path Γ from the condition $\int_\Omega q \cdot \psi = 0$ for every divergence-free ψ. It gave me the greatest

Originality. The construction of primitives, the Cauchy integral and the weighting with which they are obtained are performed for a function taking values in a *Neumann space*, that is, a space in which every Cauchy sequence converges.

Neumann spaces. The sequential completeness characterizing these spaces is the most general property of E that guarantees that the integral of a continuous function taking values in E will belong to it, see *Case where E is not a Neumann space* (§ 4.3, p. 92). This property is more general than the more commonly considered property of completeness, that is, the convergence of all Cauchy filters; for example, if E is an infinite-dimensional Hilbert space, then E-weak is a Neumann space but is not complete [Vol. 1, Property (4.11), p. 82].

Moreover, sequential completeness is more straightforward than completeness.

Semi-norms. We use families of semi-norms, instead of the equivalent notion of locally convex topologies, to be able to define differentiability (p. 73) by comparing the semi-norms of a variation of the variable to the semi-norms of the variation of the value. A section on *Familiarization with Semi-normed Spaces* can be found on p. xiii. Semi-norms can be manipulated in a similar fashion to normed spaces, except that we are working with several semi-norms instead of a single norm.

Primitives. We show that any continuous field $q = (q_1, \ldots, q_d)$ has a primitive f, namely that $\nabla f = q$, if and only if it is orthogonal to the divergence-free test fields, that is, if $\int_\Omega q \cdot \psi = 0_E$ for every $\psi = (\psi_1, \ldots, \psi_d)$ such that $\nabla \cdot \psi = 0$. This is the *orthogonality theorem* (Theorem 9.2).

When Ω is simply connected, for a primitive f to exist, it is necessary and sufficient for q to have local primitives. This is the *local primitive gluing theorem* (Theorem 9.4). On any such open set, it is also necessary and sufficient that it verifies Poincaré's condition $\partial_i q_j = \partial_j q_i$ for every i and j to be satisfied if the field is $\mathcal{C}^1$ (Theorem 9.10), or a weak version of this condition, $\int_\Omega q_j \partial_i \varphi = \int_\Omega q_i \partial_j \varphi$ for every test function φ, if the field is continuous (Theorem 9.11).

We explicitly determine all primitives (Theorem 9.17) and construct one that depends continuously on q (Theorem 9.18).

Integration. We extend the Cauchy integral to uniformly continuous functions taking values in a Neumann space, because this will be an essential tool for constructing primitives.

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mathematical satisfaction I have ever experienced to explicitly construct an incompressible tubular flow (see p. 184). Twenty-five years later, I am finally ready to answer any other questions from my (very persistent) students.

The properties established here for continuous functions will also be used to extend them to integrable distributions in Volume 4, by continuity or transposition. Indeed, one of the objectives of the *Analysis for PDEs* series is to extend integration and Sobolev spaces to take values in Neumann spaces. However, it seemed more straightforward to first construct distributions (in Volume 3) using just continuous functions before introducing integrable distributions (in Volume 4), which play the role usually fulfilled by *classes of almost everywhere equal integrable functions*.

Weighting. The weighted function $f \diamond \mu$ of a function f defined on an open set Ω by the weight μ, a real function with compact support D, is a function defined on the open set $\Omega_D = \{x \in \mathbb{R}^d : x + D \subset \Omega\}$ by $(f \diamond \mu)(x) = \int_{\mathring{D}} f(x+y)\,\mu(y)\,\mathrm{d}y$. This concept will be repeatedly useful. It plays an analogous role to convolution, which is equivalent to it up to a symmetry of μ when $\Omega = \mathbb{R}^d$.

Novelties. Many results are natural extensions of previous results, but the following seemed most noteworthy:
— The construction of the topology of the space $\mathcal{K}(\Omega; E)$ of continuous functions with compact support using the semi-norms $\|f\|_{\mathcal{K}(\Omega;E);q} = \sup_{x\in\Omega} q(x)\|f(x)\|_{E;\nu}$ indexed by $q \in \mathcal{C}^+(\Omega)$ and $\nu \in \mathcal{N}_E$ (Definition 1.17). This is equivalent to and much simpler than the inductive limit topology of the $\mathcal{C}_K(\Omega; E)$.
— The fact that if a function $f \in \mathcal{C}(\Omega)$ satisfies $\sup_{x\in\Omega} q(x)|f(x)| < \infty$ for every $q \in \mathcal{C}^+(\Omega)$, then its support is compact (Theorem 1.22). This is the basis for defining the semi-norms of $\mathcal{D}(\Omega)$ in Volume 3.
— The *concentration theorem* for the integral and the construction of an incompressible *tubular flow* (Theorems 8.18 and 8.17), which are key steps in our construction of the primitives of a field taking values in a Neumann space, as it is explained in the comment *Utility of the concentration theorem*, p. 186.

Prerequisites. The proofs in the main body of the text only use definitions and results established in Volume 1, whose statements are recalled either in the text or in the Appendix. Detailed proofs are given, including arguments that may seem trivial to experienced readers, and the theorem numbers are systematically referenced.

Comments. Comments with a smaller font size than the main body of the text appeal to external results or results that have not yet been established. The Appendix on *Reminders* is also written with a smaller font size, since its contents are assumed to be familiar.

Historical notes. Wherever possible, the origin of the concepts and results is given as a footnote[2].

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2. **Appeal to the reader.** Many important results lack historical notes because I am not familiar with their origins. I hope that my readers will forgive me for these omissions and any injustices they may discover. And I encourage the scholars among you to notify me of any improvements for future editions!

Navigation through the book:
— The **Table of Contents** at the start of the book lists the topics discussed.
— The **Table of Notations**, p. xv, specifies the meaning of the notation in case there is any doubt.
— The **Index**, p. 243, provides an alternative access to specific topics.
— All hypotheses are stated directly within the theorems themselves.
— The numbering scheme is shared across every type of statement to make results easier to find by number (for instance, Theorem 2.9 is found between the statements 2.8 and 2.10, which are a definition and a theorem, respectively).

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Olivier BESSON, Fulbert MIGNOT, Nicolas DEPAUW, and Didier BRESCH also provided many improvements, in form and in substance.

Pierre DREYFUSS gave me insight into the necessity of simply connected domains for the existence of primitives with Poincaré's condition, as explained on p. 209 in the comment *Is simple connectedness necessary for gluing together local primitives?*

Joshua PEPPER spent much time discussing about the best way to adapt this work in English.

Thank you, my friends.

Jacques SIMON
Chapdes-Beaufort,
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Chapter 8

Line Integral of a Vector Field Along a Path

This chapter provides us two essential results to construct primitives:
— The *concentration theorem* (Theorem 8.18) shows that, for any field $q = (q_1, \dots, q_d)$, the integral $\int_\Omega q \cdot \Psi$ is equal to the integral $\fint_\Gamma q \diamond \rho \cdot \mathrm{d}\ell$ around the closed path Γ, where Ψ is a divergence-free *tubular flow* constructed in Theorem 8.18 with support in a tube of axis Γ. Some applications are mentioned in the comment *Utility of the concentration theorem* (p. 186).
— The *theorem on the invariance under homotopy of the line integral of local gradients* (Theorem 8.20) shows that if a field q is of the form $q = \nabla f_B$ on every ball B, then its line integral $\fint_\Gamma q \cdot \mathrm{d}\ell$ around a closed path Γ is invariant under homotopy. Some applications are mentioned in the comment *Utility of the invariance theorem...* (p. 187).

We therefore begin by studying the line integral $\fint_\Gamma q \cdot \mathrm{d}\ell \overset{\text{def}}{=} \int_{t_\mathrm{i}}^{t_\mathrm{e}} (q \circ \Gamma) \cdot \Gamma' \,\mathrm{d}t$ (Definition 8.7) of a field $q \in \mathcal{C}(\Omega; E^d)$ along a path $\Gamma \in \mathcal{C}^1([t_\mathrm{i}, t_\mathrm{e}]; \Omega)$. In particular:
— The line integral can be concatenated (Theorem 8.14), i.e. $\fint_\Gamma = \sum_n \fint_{\Gamma_n}$ if $\Gamma = \overrightarrow{\cup}_n \Gamma_n$.
— We can reparametrize any concatenation of $\mathcal{C}^1$ paths, or in other words any piecewise $\mathcal{C}^1$ path, as a $\mathcal{C}^1$ path (Theorem 8.4), without changing the line integral (Theorem 8.16).
— The line integral of a gradient around a closed path is zero (Theorem 8.11), i.e. $\fint_\Gamma \nabla f \cdot \mathrm{d}\ell = 0$.

8.1. Paths

Let us define paths and closed paths in a separated semi-normed space.

DEFINITION 8.1.– *Let E be a separated semi-normed space and $U \subset E$.*

(a) *A **path** in U is a mapping $\Gamma \in \mathcal{C}([t_\mathrm{i}, t_\mathrm{e}]; U)$, where $[t_\mathrm{i}, t_\mathrm{e}]$ is a closed and bounded interval of $\mathbb{R}$.*

*We say that Γ **joins** the **initial point** $\Gamma(t_\mathrm{i})$ to the **ending point** (or **terminal point**) $\Gamma(t_\mathrm{e})$ in U. The **image** of Γ is the set $[\Gamma] = \{\Gamma(t) : t_\mathrm{i} \le t \le t_\mathrm{e}\}$.*

(b) *A **closed path** is a path whose initial point and ending point coincide.*

(c) *We say that a path* **is** $\mathcal{C}^1$, *or* **of class** $\mathcal{C}^1$, *if it is of the form* $\Gamma \in \mathcal{C}^1([t_\mathrm{i}, t_\mathrm{e}]; U)$.

In other words (Definition 2.26), a path is $\mathcal{C}^1$ *if the derivative* Γ', *which is initially only defined on the open set* $(t_\mathrm{i}, t_\mathrm{e})$, *has a continuous extension to* $[t_\mathrm{i}, t_\mathrm{e}]$, *still denoted by* Γ'. ▮

Geometry. The image $[\Gamma]$ of a $\mathcal{C}^1$ path is not necessarily a regular curve or a one-dimensional manifold. It may just be a single point, intersect itself, form angles (between segments, see Theorem 8.4) or cusps, and so on. □

Let us define the "reverse path," where the initial point and ending point are interchanged.

DEFINITION 8.2.– *Let* $\Gamma \in \mathcal{C}([t_\mathrm{i}, t_\mathrm{e}]; E)$ *be a path in a separated semi-normed space* E. *The* **reverse path** *of* Γ *is the path* $\overleftarrow{\Gamma}$ *defined on* $[-t_\mathrm{e}, -t_\mathrm{i}]$ *by*

$$\overleftarrow{\Gamma}(t) \stackrel{\text{def}}{=} \Gamma(-t). \quad ▮$$

Let us concatenate two paths when the ending point of the first is the initial point of the second.

DEFINITION 8.3.– *Let* $\Gamma_1 \in \mathcal{C}([t_{\mathrm{i}_1}, t_{\mathrm{e}_1}]; E)$ *and* $\Gamma_2 \in \mathcal{C}([t_{\mathrm{i}_2}, t_{\mathrm{e}_2}]; E)$ *be two paths in a separated semi-normed space* E *such that*

$$\Gamma_1(t_{\mathrm{e}_1}) = \Gamma_2(t_{\mathrm{i}_2}).$$

Their **concatenation** *is the path* $\Gamma_1 \overrightarrow{\cup} \Gamma_2$ *defined on* $[t_{\mathrm{i}_1}, t_{\mathrm{e}_1} + t_{\mathrm{e}_2} - t_{\mathrm{i}_2}]$ *by:*

$$(\Gamma_1 \overrightarrow{\cup} \Gamma_2)(t) \stackrel{\text{def}}{=} \begin{cases} \Gamma_1(t) & \text{for } t_{\mathrm{i}_1} \le t \le t_{\mathrm{e}_1}, \\ \Gamma_2(t + t_{\mathrm{i}_2} - t_{\mathrm{e}_1}) & \text{for } t_{\mathrm{e}_1} \le t \le t_{\mathrm{e}_1} + t_{\mathrm{e}_2} - t_{\mathrm{i}_2}. \end{cases} \quad ▮$$

Let us show that we can **reparametrize** any concatenation of $\mathcal{C}^1$ paths to obtain a $\mathcal{C}^1$ path.

THEOREM 8.4.– *Let* $\Gamma = \overrightarrow{\bigcup}_{1 \le n \le N} \Gamma_n$ *be the concatenation of finitely many* $\mathcal{C}^1$ *paths in* $\mathbb{R}^d$, *and let* $[t_\mathrm{i}, t_\mathrm{e}]$ *be its interval of definition.*

Then there exists a bijection $T \in \mathcal{C}^1([t_\mathrm{i}, t_\mathrm{e}])$ *from* $[t_\mathrm{i}, t_\mathrm{e}]$ *onto itself such that* T' *vanishes at the initial point and at the ending point of each* Γ_n *and is* > 0 *outside of these points. For any such bijection,*

$$\Gamma \circ T \text{ is a } \mathcal{C}^1 \text{ path.} \quad ▮$$

Proof. Let $\Lambda \in \mathcal{C}^1([a,b])$ be one of the pieces of Γ and

$$T(t) = a + (b-a)\Big(3\Big(\frac{t-a}{b-a}\Big)^2 - 2\Big(\frac{t-a}{b-a}\Big)^3\Big).$$

Its derivative $T'(t) = 6(t-a)/(b-a) - 6((t-a)/(b-a))^2$ is continuous and > 0 on (a,b), and its extension by 0 is continuous on $[a,b]$.

By Theorem 3.12 (c) on differentiating composite functions, $\Lambda \circ T$ is differentiable and $(\Lambda \circ T)'(t) = (\Lambda' \circ T)(t)\, T'(t)$. This expression tends to 0 as $t \to a$ or $t \to b$, since $(\Lambda' \circ T)(t)$ remains bounded and $T'(t) \to 0$.

By reparametrizing each piece Γ_n of Γ in this way, we obtain a function $\Gamma \circ T$ that is continuous on $[t_\mathrm{i}, t_\mathrm{e}]$, differentiable outside of the points joining different pieces together, and whose derivative tends to 0 at each of these points. By Theorem 2.28 on extending the derivative, it follows that $\Gamma \circ T$ is differentiable at these points and continuously differentiable on $(t_\mathrm{i}, t_\mathrm{e})$. The extension by 0 of its derivative is continuous on the whole of $[t_\mathrm{i}, t_\mathrm{e}]$; in other words, $\Gamma \circ T \in \mathcal{C}^1([t_\mathrm{i}, t_\mathrm{e}]; E)$. □

Let us show that connected open sets are path connected.

THEOREM 8.5.– *Any pair of points of a connected open subset U of separated semi-normed space can be joined by a $\mathcal{C}^1$ path in U.* ▮

Proof. Let E be the space in question, $a \in U$, and X the set of points of U that can be joined to a by a $\mathcal{C}^1$ path in U. It must be proved that $X = U$.

Let us first show that X is open. Let $x \in X$ and $\{\| \, \|_{E;\nu} : \nu \in \mathcal{N}_E\}$ the family of semi-norms of E. By Definition A.7 (b) of an open set, here U, there exists a finite subset N of $\mathcal{N}_E$ and $\epsilon > 0$ such that the ball $B = \{v \in E : \sup_{\nu \in N} \|v - x\|_{E;\nu} \leq \epsilon\}$ is included in U. Since x can be joined to a by a path Γ in U of class $\mathcal{C}^1$, every point v of B can be joined to a by the concatenation of Γ and the line segment $[x, v]$. By reparametrizing this concatenation using Theorem 8.4, we obtain a $\mathcal{C}^1$ path that joins v to x. Therefore, $B \subset X$, which shows that X is open in E.

Similarly, its complement $Y = U \setminus X$ is open. Indeed, if we now assume that $x \in Y$, no point of B can be joined to a by a $\mathcal{C}^1$ path in U, otherwise it would also be possible for x, so $B \subset Y$.

The open sets X and Y are disjoint, cover the connected set U and X is non-empty (since it contains a). Therefore, by Definition A.15 of a connected set, $Y = \emptyset$, and $X = U$. □

Path connected. A set is said to be **path connected** if any two of its points can be joined by a path. Theorem 8.5 is slightly stronger, since it gives us $\mathcal{C}^1$ paths. It would also be possible to construct $\mathcal{C}^\infty$ paths, but this is not necessary for our purposes. □

Let us show that, conversely, any two points joined by a path belong to the same connected component.

THEOREM 8.6.– *If two points are connected by a path in a subset U of a separated semi-normed space, then they belong to the same connected component of U.* ▮

Proof. Let E be the space in question and Γ a path in U joining two points a and b. By Definition 8.1, the image $[\Gamma]$ of this path is the image of an interval $[t_\mathrm{i}, t_\mathrm{e}]$ under the continuous mapping Γ. Every interval being connected (Theorem A.16), $[\Gamma]$ is therefore connected (Theorem A.33). Furthermore, it is included in U (by the hypotheses) and contains $\Gamma(t_\mathrm{i})$, that is, a.

By Definition A.15, the connected component generated by a is the largest connected set included in U that contains a. It therefore contains $[\Gamma]$ and certainly also contains $\Gamma(t_\mathrm{e})$, that is, b. □

8.2. Line integral of a field along a path

A **vector field** on a subset of $\mathbb{R}^d$ is any function with d components taking values in a space E, or equivalently any function taking values in the Euclidean product E^d.

Let us define the line integral[1] along a $\mathcal{C}^1$ path of a vector field taking values in a Neumann space.

DEFINITION 8.7.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where $\Omega \subset \mathbb{R}^d$ and E is a Neumann space, and let $\Gamma \in \mathcal{C}^1([t_\mathrm{i}, t_\mathrm{e}]; \Omega)$ be a path in Ω. We denote by Γ' the derivative of Γ.*

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1. **History of the line integral of a field along a path.** The line integral of a vector field along a path was introduced by Gaspard-Gustave DE CORIOLIS in 1829 [26] to express the work of a force, i.e. the variation of the kinetic energy of a body that moves under the action of this force.
This notion was developed as part of the theory of differential forms established around 1890–1900 by Émile CARTAN [18, Vol. II, pp. 309–396] and Henri POINCARÉ [63, Vol. III, Chapter XXII]; see, for example, [CARTAN, Henri, 19, pp. 215–219] (the son of Émile whom we mentioned above), where the results of the sections 8.2 and 8.3 of the present book can be found with the field q "hidden" behind the 0-form ω and the gradient ∇f "hidden" behind the 1-form ω or $\mathrm{d}g$.

French terminology. In French, the line integral is called the *circulation*, a term that English-speakers reserve for the case where the path is closed. The French term *intégrale curviligne*, which is the word-for-word translation of "line integral," is generally reserved for the line integral of a scalar function.

*The **line integral** of q along Γ is the element of E given by*

$$\int_{\Gamma} q \cdot \mathrm{d}\ell \stackrel{\text{def}}{=} \int_{t_\mathrm{i}}^{t_\mathrm{e}} (q \circ \Gamma) \cdot \Gamma' \, \mathrm{d}t. \;\blacksquare$$

Inconsistent notation! If we want to be consistent with our notation for the Cauchy integral, we need to write either $\int_{t_\mathrm{i}}^{t_\mathrm{e}} (q \circ \Gamma)(t) \cdot \Gamma'(t) \, \mathrm{d}t$ or $\int_{t_\mathrm{i}}^{t_\mathrm{e}} (q \circ \Gamma) \cdot \Gamma'$ here. However, we will add $\mathrm{d}t$ to the latter anyway to mirror the usage of $\mathrm{d}\ell$. $\square$

Justification of Definition 8.7. For the right-hand side to be defined, by Definition 4.9 of the integral taking values in a Neumann space, the function $(q \circ \Gamma) \cdot \Gamma'$ must be uniformly continuous on $(t_\mathrm{i}, t_\mathrm{e})$.

To check this, observe that the composite mapping $q \circ \Gamma$ is continuous (Theorem A.35) on $[t_\mathrm{i}, t_\mathrm{e}]$, as well as Γ' (extended as in Definition 8.1 (c)). Their product $(q \circ \Gamma) \cdot \Gamma'$ is therefore continuous (Theorem A.35 again) because it is obtained by composing them with the mapping $\cdot$, which is continuous by the inequality (2.2) (p. 31). Therefore, by Heine's theorem (Theorem A.34), $(q \circ \Gamma) \cdot \Gamma'$ is uniformly continuous on the compact set $[t_\mathrm{i}, t_\mathrm{e}]$ and thus on $(t_\mathrm{i}, t_\mathrm{e})$.

It is also necessary for $(q \circ \Gamma) \cdot \Gamma'$ to have bounded support. This is the case because $[t_\mathrm{i}, t_\mathrm{e}]$ is bounded by Definition 8.1 (a) of a path. $\square$

Let us show that the sign of the line integral changes when the path is reversed.

THEOREM 8.8.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where $\Omega \subset \mathbb{R}^d$ and E is a Neumann space, and let Γ be a $\mathcal{C}^1$ path in Ω. Then*

$$\int_{\overleftarrow{\Gamma}} q \cdot \mathrm{d}\ell = -\int_{\Gamma} q \cdot \mathrm{d}\ell. \;\blacksquare$$

Proof. By Definition 8.2 of the reverse path and Definition 8.7 of the line integral, since $\mathrm{d}(\Gamma(-t))/\mathrm{d}t = -(\mathrm{d}\Gamma/\mathrm{d}t)(-t)$,

$$\int_{\overleftarrow{\Gamma}} q \cdot \mathrm{d}\ell = \int_{-t_\mathrm{e}}^{-t_\mathrm{i}} q \circ \Gamma(-t) \cdot \frac{\mathrm{d}(\Gamma(-t))}{\mathrm{d}t} \, \mathrm{d}t = -\int_{-t_\mathrm{e}}^{-t_\mathrm{i}} \Big(q \circ \Gamma \cdot \frac{\mathrm{d}\Gamma}{\mathrm{d}t} \Big)(-t) \, \mathrm{d}t.$$

The integral being invariant under a symmetry by Theorem 6.18, we obtain

$$\int_{\overleftarrow{\Gamma}} q \cdot \mathrm{d}\ell = -\int_{t_\mathrm{i}}^{t_\mathrm{e}} \Big(q \circ \Gamma \cdot \frac{\mathrm{d}\Gamma}{\mathrm{d}t} \Big)(t) \, \mathrm{d}t = -\int_{\Gamma} q \cdot \mathrm{d}\ell. \;\square$$

Let us show that the line integral is invariant under an increasing change of variables.

THEOREM 8.9.– *Let Γ be a $\mathcal{C}^1$ path in a subset Ω of $\mathbb{R}^d$, defined on a bounded interval $[t_\mathrm{i}, t_\mathrm{e}]$, and let T be a bijection from a bounded interval $[t'_\mathrm{i}, t'_\mathrm{e}]$ onto $[t_\mathrm{i}, t_\mathrm{e}]$ such that:*

$$T \in \mathcal{C}^1([t'_\mathrm{i}, t'_\mathrm{e}]), \;\; T' > 0 \text{ on } (t'_\mathrm{i}, t'_\mathrm{e}).$$

Then $\Gamma \circ T$ is a $\mathcal{C}^1$ path in Ω and, for every $q \in \mathcal{C}(\Omega; E^d)$, where E is a Neumann space,

$$\int_{\Gamma \circ T} q \cdot \mathrm{d}\ell = \int_\Gamma q \cdot \mathrm{d}\ell. \;\blacksquare$$

Proof. Let us first check that $\Gamma \circ T$ is a $\mathcal{C}^1$ path. By Theorem 3.12 (c) on differentiating composite functions, on $(t_\mathrm{i}, t_\mathrm{e})$,

$$(\Gamma \circ T)' = (\Gamma' \circ T)\, T'. \tag{8.1}$$

The right-hand side, and hence the left-hand side, is uniformly continuous because Γ', T and T' are uniformly continuous by the hypotheses, and therefore so are $\Gamma' \circ T$ (Theorem A.35) and its product with T' (Theorem 3.5 (b)). It therefore has (Theorem A.38) a continuous extension on $[t'_\mathrm{i}, t'_\mathrm{e}]$. By Definition 8.1 (c) of a $\mathcal{C}^1$ path, this proves that $\Gamma \circ T \in \mathcal{C}^1([t'_\mathrm{i}, t'_\mathrm{e}]; \mathbb{R}^d)$.

Let us now prove the invariance of the line integral. Its Definition 8.7 gives, together with (8.1),

$$\int_{\Gamma \circ T} q \cdot \mathrm{d}\ell = \int_{t'_\mathrm{i}}^{t'_\mathrm{e}} (q \circ \Gamma \circ T) \cdot (\Gamma \circ T)' \,\mathrm{d}t = \int_{t'_\mathrm{i}}^{t'_\mathrm{e}} (((q \circ \Gamma) \cdot \Gamma') \circ T)\, T' \,\mathrm{d}t.$$

Transforming the latter expression with the change of variables formula for an integral from Theorem 6.14 (c), where in this case $|\det[\nabla T]| = T'$ (because $\nabla T = T'$, which is positive by the hypotheses) gives

$$\int_{\Gamma \circ T} q \cdot \mathrm{d}\ell = \int_{t_\mathrm{i}}^{t_\mathrm{e}} (q \circ \Gamma) \cdot \Gamma' \,\mathrm{d}t = \int_\Gamma q \cdot \mathrm{d}\ell.$$

It remains to be checked that T^{-1} is $\mathcal{C}^1$ because this is assumed by Theorem 6.14 (c). Since $T' > 0$, Theorem A.55 on differentiating the inverse of a function implies that T^{-1} is continuous and differentiable and $(T^{-1})'(t) = 1/(T'(T^{-1}(t)))$. In other words, $(T^{-1})' = \mathcal{Q} \circ T' \circ T^{-1}$, where $\mathcal{Q}(x) = 1/x$. Therefore, $(T^{-1})'$ is continuous, like any composition of continuous mappings (Theorem A.35), since, in addition to T' and T^{-1}, $\mathcal{Q}$ is also continuous (from $(0, \infty)$ into $\mathbb{R}$, by Theorem A.56). $\square$

Independence of the parametrization. By Theorems 8.8 and 8.9, the line integral along a path Γ of class $\mathcal{C}^1$ only depends on its geometry (in other words, its image $[\Gamma]$) and the direction along which it is integrated. $\square$

Line integral along a curve. If the image $[\Gamma]$ of Γ is a rectifiable curve,

$$\int_{\Gamma} q \cdot \mathrm{d}\ell = \int_{[\Gamma]} q \cdot \tau \, \mathrm{d}\sigma,$$

where $\mathrm{d}\sigma$ is the infinitesimal arc length of $[\Gamma]$ and τ is the oriented unit tangent vector, i.e. $\tau = \Gamma'/|\Gamma'|$ whenever Γ is a $\mathcal{C}^1$ injection such that Γ' does not vanish. $\square$

Let us calculate the line integral along a path consisting of a single point or along a rectilinear path.

THEOREM 8.10.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where $\Omega \subset \mathbb{R}^d$ and E is a Neumann space, and let a and x be two points of Ω such that $[a, x] \subset \Omega$. Then:*

(a) *If $\Gamma_{\{a\}}$ is the path defined on $[0, 1]$ by $\Gamma_{\{a\}}(t) = a$,*

$$\int_{\Gamma_{\{a\}}} q \cdot \mathrm{d}\ell = 0_E.$$

(b) *If $\Gamma_{\overrightarrow{a,x}}$ is the **rectilinear path** defined on $[0, 1]$ by $\Gamma_{\overrightarrow{a,x}}(t) = a + t(x - a)$,*

$$\int_{\Gamma_{\overrightarrow{a,x}}} q \cdot \mathrm{d}\ell = (x - a) \cdot \int_0^1 q(a + t(x - a))\, \mathrm{d}t. \blacksquare$$

Proof. Apply Definition 8.7 of the line integral and the equalities $\mathrm{d}a/\mathrm{d}t = 0$ and $\mathrm{d}(a + t(x - a))/\mathrm{d}t = x - a$, respectively. $\square$

Let us calculate the **line integral of a gradient**.

THEOREM 8.11.– *Let $f \in \mathcal{C}^1(\Omega; E)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space, and let Γ be a $\mathcal{C}^1$ path in Ω. Then:*

(a) *If a is the initial point of Γ and b is its ending point,*

$$\int_{\Gamma} \nabla f \cdot \mathrm{d}\ell = f(b) - f(a).$$

(b) *If Γ is a closed path,*

$$\int_{\Gamma} \nabla f \cdot \mathrm{d}\ell = 0_E. \blacksquare$$

Proof. (a) By Theorem 3.12 (a) on changes of variables in a derivative with $\ell = 1$ and $\partial_i = \mathrm{d}/\mathrm{d}t$,

$$(f \circ \Gamma)' = \sum_{j=1}^{d} (\partial_j f \circ \Gamma)\, \Gamma_j' = (\nabla f \circ \Gamma) \boldsymbol{\cdot} \Gamma'.$$

Definition 8.7 of the line integral therefore gives

$$\int_\Gamma \nabla f \boldsymbol{\cdot} \mathrm{d}\ell = \int_{t_\mathrm{i}}^{t_\mathrm{e}} (\nabla f \circ \Gamma) \boldsymbol{\cdot} \Gamma' \,\mathrm{d}t = \int_{t_\mathrm{i}}^{t_\mathrm{e}} (f \circ \Gamma)' \,\mathrm{d}t.$$

With Theorem 6.4 (b) on calculating the integral of a derivative, this gives

$$\int_\Gamma \nabla f \boldsymbol{\cdot} \mathrm{d}\ell = (f \circ \Gamma)(t_\mathrm{e}) - (f \circ \Gamma)(t_\mathrm{i}) = f(b) - f(a).$$

(b) This follows from (a) because the ending point of a closed path coincides with its initial point by Definition 8.1 (b). □

Let us show that the line integral of a vector field depends continuously on the vector field.

THEOREM 8.12.– *Let $\Omega \subset \mathbb{R}^d$, E a Neumann space and Γ a $\mathcal{C}^1$ path in Ω. Then:*

(a) *For every $q \in \mathcal{C}(\Omega; E^d)$ and every semi-norm $\|\ \|_{E;\nu}$ of E,*

$$\Big\| \int_\Gamma q \boldsymbol{\cdot} \mathrm{d}\ell \Big\|_{E;\nu} \le \gamma\, |t_\mathrm{e} - t_\mathrm{i}| \sup_{x \in [\Gamma]} \|q(x)\|_{E^d;\nu},$$

where $[\Gamma] = \{\Gamma(t) : t_\mathrm{i} < t < t_\mathrm{e}\}$ and $\gamma = \sup_{t_\mathrm{i} < t < t_\mathrm{e}} |\Gamma'(t)| < \infty$.

(b) *The mapping $q \mapsto \int_\Gamma q \boldsymbol{\cdot} \mathrm{d}\ell$ is linear and continuous from $\mathcal{C}(\Omega; E^d)$ into E.* ▮

Proof. (a) Definition 8.7 of the line integral and the bound on the semi-norms of the integral from Theorem 4.15 give

$$\Big\| \int_\Gamma q \boldsymbol{\cdot} \mathrm{d}\ell \Big\|_{E;\nu} = \Big\| \int_{t_\mathrm{i}}^{t_\mathrm{e}} (q \circ \Gamma) \boldsymbol{\cdot} \Gamma' \,\mathrm{d}t \Big\|_{E;\nu} \le$$
$$\le |t_\mathrm{e} - t_\mathrm{i}| \sup_{t_\mathrm{i} < t < t_\mathrm{e}} \|((q \circ \Gamma) \boldsymbol{\cdot} \Gamma')(t)\|_{E;\nu} \le \gamma\, |t_\mathrm{e} - t_\mathrm{i}| \sup_{x \in [\Gamma]} \|q(x)\|_{E^d;\nu},$$

where $\gamma = \sup_{t_\mathrm{i} < t < t_\mathrm{e}} |\Gamma'(t)|$. This quantity is finite since, by Definition 8.1 (c) of a $\mathcal{C}^1$ path, Γ' may be continuously extended to $[t_\mathrm{i}, t_\mathrm{e}]$, and since a continuous function on a compact set is bounded (Theorem A.34).

(b) By Definition 1.3 (a) of the semi-norms of $\mathcal{C}(\Omega; E^d)$, the above inequality can be stated as

$$\Big\| \int_\Gamma q \cdot \mathrm{d}\ell \Big\|_{E;\nu} \leq c \sup_{x\in[\Gamma]} \|q(x)\|_{E^d;\nu} = c\,\|q\|_{\mathcal{C}(\Omega;E^d);[\Gamma],\nu},$$

which implies the desired continuity by the characterization of continuous linear mappings from Theorem 1.25. □

8.3. Line integral along a concatenation of paths

Let us define the notion of a piecewise $\mathcal{C}^1$ path.

DEFINITION 8.13.– *A* ***piecewise*** *$\mathcal{C}^1$ path is a concatenation of finitely many $\mathcal{C}^1$ paths.* ▮

Let us first show that the line integral along a $\mathcal{C}^1$ concatenation of paths is the sum of the line integrals along each piece.

THEOREM 8.14.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where $\Omega \subset \mathbb{R}^d$ and E is a Neumann space, and let Γ, Γ_1, ..., and Γ_N be $\mathcal{C}^1$ paths in Ω such that*

$$\Gamma = \overrightarrow{\bigcup_{1\leq n\leq N}} \Gamma_n.$$

Then

$$\int_\Gamma q \cdot \mathrm{d}\ell = \sum_{1\leq n\leq N} \int_{\Gamma_n} q \cdot \mathrm{d}\ell. \;▮$$

Proof. By Definition 8.7 of the line integral, we have to show that

$$\int_{t_\mathrm{i}}^{t_\mathrm{e}} (q\circ\Gamma)\cdot\Gamma' \,\mathrm{d}t = \sum_{1\leq n\leq N} \int_{t_{\mathrm{i}_n}}^{t_{\mathrm{e}_n}} (q\circ\Gamma)\cdot\Gamma' \,\mathrm{d}t.$$

It follows from the additivity with respect to the interval of integration (Theorem 6.2) since, by Definition 8.3 of concatenation, $t_\mathrm{i} = t_{\mathrm{i}_1} \ldots < t_{\mathrm{e}_n} = t_{\mathrm{i}_{n+1}} < \ldots t_{\mathrm{e}_N} = t_\mathrm{e}$. □

Let us now extend this property to piecewise $\mathcal{C}^1$ paths that are not necessarily $\mathcal{C}^1$ as a whole by using it as the definition of the line integral along such a path.

DEFINITION 8.15.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where $\Omega \subset \mathbb{R}^d$ and E is a Neumann space, and consider a piecewise $\mathcal{C}^1$ path in Ω*

$$\Gamma = \overrightarrow{\bigcup_{1 \le n \le N}} \Gamma_n.$$

*The **line integral** of q along Γ is here the element of E defined by*

$$\fint_\Gamma q \cdot \mathrm{d}\ell \stackrel{\mathrm{def}}{=} \sum_{1 \le n \le N} \fint_{\Gamma_n} q \cdot \mathrm{d}\ell. \blacksquare$$

Justification. The notation $\fint_\Gamma$ is admissible because, when Γ is $\mathcal{C}^1$, it has the same line integral as in Definition 8.7 by the additivity property of Theorem 8.14.

This definition is still admissible for piecewise $\mathcal{C}^1$ paths because the line integral does not depend on how Γ is partitioned into $\mathcal{C}^1$ pieces, even though there are infinitely many possible partitions. Indeed, it is equal to the line integral of the *minimal* partition, which is unique. More precisely, define $t_1 = t_\mathrm{i}$, then define t_{i+1} inductively as the largest real number such that the restriction Λ_i of Γ to $[t_i, t_{i+1}]$ is a $\mathcal{C}^1$ path. Repeat until $t_{I+1} = t_\mathrm{e}$. Then

$$\Gamma = \overrightarrow{\bigcup_{1 \le i \le I}} \Lambda_i.$$

This partition, called the *minimal* partition, only depends on Γ and not on the original partition into Γ_n. We indeed find the line integral of this minimal partition from the above definition, since, for every $i \in [\![1, I]\!]$, there exists $n_i \in \mathbb{N}$ such that $\Lambda_i = \overrightarrow{\bigcup}_{n_i \le n < n_{i+1}} \Gamma_n$, and therefore, again by the additivity property of Theorem 8.14,

$$\sum_{1 \le n \le N} \fint_{\Gamma_n} q \cdot \mathrm{d}\ell = \sum_{1 \le i \le I} \sum_{n_i \le n < n_{i+1}} \fint_{\Gamma_n} q \cdot \mathrm{d}\ell = \sum_{1 \le i \le I} \fint_{\Lambda_i} q \cdot \mathrm{d}\ell. \ \square$$

Let us show reparametrizing a piecewise $\mathcal{C}^1$ path as a $\mathcal{C}^1$ path does not change the line integral.

THEOREM 8.16.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where $\Omega \subset \mathbb{R}^d$ and E is a Neumann space, and consider a piecewise $\mathcal{C}^1$ path in Ω*

$$\Gamma = \overrightarrow{\bigcup_{1 \le n \le N}} \Gamma_n.$$

Let T be a reparametrization of Γ as a $\mathcal{C}^1$ path, that is, let T be a bijection from the interval on which Γ is defined onto itself as given by Theorem 8.4. Then

$$\fint_{\Gamma \circ T} q \cdot \mathrm{d}\ell = \fint_\Gamma q \cdot \mathrm{d}\ell$$

and, for every $n \in [\![1, N]\!]$,

$$\int_{\Gamma_n \circ T} q \cdot \mathrm{d}\ell = \int_{\Gamma_n} q \cdot \mathrm{d}\ell. \blacksquare$$

Proof. For each piece Γ_n of Γ, Theorem 8.9 on changes of variables in the line integral along a $\mathcal{C}^1$ path gives, since T is $\mathcal{C}^1$ on $[t_{\mathrm{i}_n}, t_{\mathrm{e}_n}]$ and $T' > 0$ on $(t_{\mathrm{i}_n}, t_{\mathrm{e}_n})$,

$$\int_{\Gamma_n \circ T} q \cdot \mathrm{d}\ell = \int_{\Gamma_n} q \cdot \mathrm{d}\ell.$$

Since $\Gamma \circ T$ is the concatenation of the $\Gamma_n \circ T$, Definition 8.15 of the line integral along a piecewise $\mathcal{C}^1$ path gives

$$\int_{\Gamma \circ T} q \cdot \mathrm{d}\ell = \sum_n \int_{\Gamma_n \circ T} q \cdot \mathrm{d}\ell = \sum_n \int_{\Gamma_n} q \cdot \mathrm{d}\ell = \int_{\Gamma} q \cdot \mathrm{d}\ell. \ \square$$

8.4. Tubular flow and the concentration theorem

Let us construct a divergence-free test field with support in a tubular neighborhood of a path[2].

By definition, the **divergence**[3] of ψ is $\nabla \cdot \psi = \partial_1 \psi_1 + \cdots \partial_d \psi_d$.

THEOREM 8.17.– *Let* $\mathcal{T} = [\Gamma] + B$*, known as a* ***tube****, where* $\Gamma \in \mathcal{C}^1([t_\mathrm{i}, t_\mathrm{e}]; \mathbb{R}^d)$ *is a closed path in* $\mathbb{R}^d$*,* $[\Gamma] = \{\Gamma(t) : t_\mathrm{i} \le t \le t_\mathrm{e}\}$ *is its image and* B *is a compact subset of* $\mathbb{R}^d$*. Additionally, let* $\rho \in \mathcal{C}^\infty_B(\mathbb{R}^d)$*.*

We define $\Psi \in \mathcal{C}^\infty_{\mathcal{T}}(\mathbb{R}^d; \mathbb{R}^d)$*, known as the* ***tubular flow****, by*

$$\Psi(x) \stackrel{\mathrm{def}}{=} \int_{t_\mathrm{i}}^{t_\mathrm{e}} \rho(x - \Gamma(t))\, \Gamma'(t)\, \mathrm{d}t.$$

It satisfies

$$\nabla \cdot \Psi = 0. \blacksquare$$

2. **History of the construction of a tubular flow.** The divergence-free field Ψ from Theorem 8.17 was obtained by Jacques SIMON in 1993 [70, Lemma, p. 1170] by constructing a concentrated incompressible flow $\vec{\delta}_\Gamma$ and then regularizing it, as explained in the comment *Underlying idea: the concentrated flow* on the next page.
The concentrated incompressible field was also constructed by Stanislav Konstantinovitch SMIRNOV in 1993 [76, p. 842] to conversely decompose any incompressible field ψ into an integral $\psi = \int_\mu \vec{\delta}_{\Gamma_\mu} \mathrm{d}\mu$ of concentrated fields.

3. **History of the divergence.** The term *divergence* was introduced by William Kingdon CLIFFORD in 1878 [24].

Terminology. We speak of a **tubular flow** because Ψ is the velocity field of an incompressible flow (meaning that the divergence is zero) with support in the tube $\mathcal{T}$ of axis $[\Gamma]$ (see Figure 8.1). This flow is stationary outside of $\mathcal{T}$ and the flux through any section S of $\mathcal{T}$ with the same orientation as Γ is equal to 1. $\square$

Utility of tubular flows. Constructing such a flow is a key step in our construction of primitives, through the *concentration theorem* (Theorem 8.18), as it is explained in the comment *Utility* ..., p. 186. $\square$

Underlying idea: the concentrated flow. The function Ψ is the regularized function $\vec{\delta}_\Gamma \diamond \rho$ of the distribution $\vec{\delta}_\Gamma \in \mathcal{D}'(\mathbb{R}^d; \mathbb{R}^d)$ of support $[\Gamma]$ defined, for every $\phi \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^d)$, by

$$\langle \vec{\delta}_\Gamma, \phi \rangle = \fint_\Gamma \phi \cdot \mathrm{d}\ell.$$

This distribution represents a "concentrated" incompressible flow on $[\Gamma]$. If $[\Gamma]$ is a regular curve, then, at each point of Γ, the "concentrated vector" $\vec{\delta}_\Gamma$ is "equal" to the tangent vector with the same orientation as Γ.

This distribution $\vec{\delta}_\Gamma$ is divergence free, namely $\nabla \cdot \vec{\delta}_\Gamma = 0$, because, for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle \nabla \cdot \vec{\delta}_\Gamma, \varphi \rangle = -\langle \vec{\delta}_\Gamma, \nabla\varphi \rangle = -\fint_\Gamma \nabla\varphi \cdot \mathrm{d}\ell = 0,$$

since the line integral of a gradient around a closed path is always zero (Theorem 8.11 (b)). Therefore, the field $\Psi = \vec{\delta}_\Gamma \diamond \rho$ is also divergence free, since

$$\nabla \cdot \Psi = \nabla \cdot (\vec{\delta}_\Gamma \diamond \rho) = (\nabla \cdot \vec{\delta}_\Gamma) \diamond \rho = 0. \ \square$$

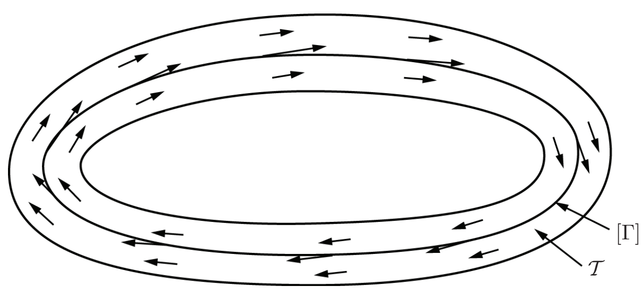

Figure 8.1. *Divergence-free tubular flow in the tube $\mathcal{T}$ of axis $[\Gamma]$*

Proof of Theorem 8.17. *Regularity of Ψ.* Its definition can be written as

$$\Psi(x) = L(R(x)),$$

where, for every $g \in \mathcal{C}(\mathbb{R}^d)$,

$$L(g) \overset{\text{def}}{=} \int_{t_\mathrm{i}}^{t_\mathrm{e}} g(-\Gamma(t))\,\Gamma'(t)\,\mathrm{d}t,$$

and where $R(x)(y) = \rho(x+y)$, i.e. $R(x) = \tau_{-x}\rho$, where τ_x is a translation.

By the bound on the semi-norms of the integral from Theorem 4.15 and Definition 1.3 (b) of the semi-norms (in this case just the norm) of $\mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)$,

$$|L(g)| \leq |t_{\mathrm{e}} - t_{\mathrm{i}}| \sup_{y \in \mathbb{R}^d} |g(y)| \sup_{t_{\mathrm{i}} \leq t \leq t_{\mathrm{e}}} |\Gamma'(t)| = \gamma \, \|g\|_{\mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)},$$

where γ only depends on Γ. By the characterization of continuous linear mappings from Theorem 1.25, this implies that $L \in \mathcal{L}(\mathcal{C}_{\mathrm{b}}(\mathbb{R}^d); \mathbb{R}^d)$. But $R \in \mathcal{C}^\infty(\mathbb{R}^d; \mathcal{C}_{\mathrm{b}}(\mathbb{R}^d))$ by the differentiability properties of the translation from Theorem 3.18 (d), since $\rho \in \mathcal{K}^\infty(\mathbb{R}^d)$ by the hypotheses. The composite mapping $L \circ R$, namely Ψ, therefore (Theorem 3.2) belongs to $\mathcal{C}^\infty(\mathbb{R}^d; \mathbb{R}^d)$.

Support of Ψ. If $x \notin [\Gamma] + B$, then, for every $t \in [t_{\mathrm{i}}, t_{\mathrm{e}}]$, we have $x - \Gamma(t) \notin B$, so $\rho(x - \Gamma(t)) = 0$, and hence $\Psi(x) = 0$. The support of Ψ is therefore included in the tube $\mathcal{T} = [\Gamma] + B$, which is compact (as a sum of compact subsets of $\mathbb{R}^d$, see Theorem A.24).

Divergence of Ψ. Let $x \in \mathbb{R}^d$. Since each mapping L_i is continuous and linear from $\mathcal{C}_{\mathrm{b}}(\mathbb{R}^d)$ into $\mathbb{R}$, it commutes with the partial derivative ∂_i by Theorem 3.1, so

$$\sum_{i=1}^{d} \partial_i \Psi_i(x) = \sum_{i=1}^{d} \partial_i (L_i(R(x))) = \sum_{i=1}^{d} L_i(\partial_i(R(x))) =$$
$$= \int_{t_{\mathrm{i}}}^{t_{\mathrm{e}}} \sum_{i=1}^{d} \partial_i \rho(x - \Gamma(t))\, \Gamma_i'(t)\, \mathrm{d}t = \int_{t_{\mathrm{i}}}^{t_{\mathrm{e}}} \nabla r(\Gamma(t)) \cdot \Gamma'(t)\, \mathrm{d}t,$$

where $r(y) = -\rho(x - y)$. The right-hand side is Definition 8.7 of the line integral of ∇r around the closed path Γ, so, by Theorem 8.11 (b), it is zero. In other words,

$$(\nabla \cdot \Psi)(x) = \fint_{\Gamma} \nabla r \cdot \mathrm{d}\ell = 0. \ \square$$

Let us show that, for any field q, the integral $\int_\Omega q \cdot \Psi$ is equal to the "concentrated" integral along Γ, $\fint_\Gamma q \diamond \rho \cdot \mathrm{d}\ell$. We call this result the **concentration theorem**[4].

THEOREM 8.18.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space.*

Let $\mathcal{T} = [\Gamma] + B$ be a tube included in Ω, where Γ is a closed $\mathcal{C}^1$ path in Ω and B is a compact subset of $\mathbb{R}^d$, $\rho \in \mathcal{C}_B^\infty(\mathbb{R}^d)$, and $\Psi \in \mathcal{C}_{\mathcal{T}}^\infty(\mathbb{R}^d; \mathbb{R}^d)$ is the tubular flow given by Theorem 8.17. Then

$$\int_\Omega q(x) \cdot \Psi(x)\, \mathrm{d}x = \fint_\Gamma q \diamond \rho \cdot \mathrm{d}\ell. \ \blacksquare$$

4. **History of the concentration theorem.** Theorem 8.18 was established for a Banach space E by Jacques SIMON in 1993 [72, p. 207, last equality].

Utility of the concentration theorem. Theorem 8.18 is a *key step* of our proof of the *orthogonality theorem* (Theorem 9.2), namely the construction of a primitive of a field q taking values in a Neumann space that is orthogonal to divergence-free test fields. Indeed, the concentration theorem is used to deduce the condition $\oint_\Gamma q \cdot \mathrm{d}\ell = 0_E$ for any closed path Γ from the orthogonality condition $\int_\Omega q \cdot \psi = 0_E$, enabling us to explicitly construct a primitive, (see the equality (9.4), p. 195). □

Proof of Theorem 8.18. By permuting the variables with Theorem 6.5, the definition of Ψ gives

$$\int_\Omega q(x) \cdot \Psi(x)\,\mathrm{d}x = \int_\Omega \sum_{i=1}^d q_i(x) \Big(\int_{t_\mathrm{i}}^{t_\mathrm{e}} \rho(x - \Gamma(t))\, \Gamma_i'(t)\,\mathrm{d}t \Big)\,\mathrm{d}x =$$

$$= \int_{t_\mathrm{i}}^{t_\mathrm{e}} \sum_{i=1}^d \Big(\int_\Omega q_i(x)\, \rho(x - \Gamma(t))\,\mathrm{d}x \Big) \Gamma_i'(t)\,\mathrm{d}t.$$

In other words, together with the expression of the weighted function from Theorem 7.2 (c) and Definition 8.7 of the line integral,

$$\int_\Omega q(x) \cdot \Psi(x)\,\mathrm{d}x = \int_{t_\mathrm{i}}^{t_\mathrm{e}} \sum_{i=1}^d (q_i \diamond \rho)(\Gamma(t))\, \Gamma_i'(t)\,\mathrm{d}t =$$

$$= \int_{t_\mathrm{i}}^{t_\mathrm{e}} (q \diamond \rho)(\Gamma(t)) \cdot \Gamma'(t)\,\mathrm{d}t = \oint_\Gamma q \diamond \rho \cdot \mathrm{d}\ell. \ \square$$

8.5. Invariance under homotopy of the line integral of a local gradient

Let us define the notion of homotopy.

DEFINITION 8.19.– *Let U be a subset of a separated semi-normed space.*

*Two closed paths Γ and Γ_* in U defined on the same interval $[t_\mathrm{i}, t_\mathrm{e}]$ are **homotopic** in U if we can transform one into the other by means of a continuous deformation. In other words, if there exists $H \in \mathcal{C}([t_\mathrm{i}, t_\mathrm{e}] \times [0,1]; U)$ such that, for every $t \in [t_\mathrm{i}, t_\mathrm{e}]$ and $s \in [0,1]$,*

$$H(t,0) = \Gamma(t), \quad H(t,1) = \Gamma_*(t), \quad H(t_\mathrm{i}, s) = H(t_\mathrm{e}, s).$$

*The **image** of H is the set $[H] = \{H(t,s) : t_\mathrm{i} \le t \le t_\mathrm{e},\ 0 \le s \le 1\}$.* ▮

Let us show that, if a field is locally a gradient, its line integral around closed paths is invariant under homotopy. We call this result the **theorem on the invariance under homotopy of the line integral of a local gradient**[5].

5. **History of the theorem on the invariance under homotopy of the line integral of a local gradient.** We are not familiar with the origin of Theorem 8.20. It is a classical result of the theory of differential

THEOREM 8.20.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space such that, for every open ball $B \Subset \Omega$, there exists $f_B \in \mathcal{C}^1(B; E)$ satisfying:*

$$\nabla f_B = q \text{ on } B.$$

Then, if Γ and Γ_* are two closed $\mathcal{C}^1$ paths that are homotopic in Ω,

$$\oint_\Gamma q \cdot \mathrm{d}\ell = \oint_{\Gamma_*} q \cdot \mathrm{d}\ell. \blacksquare$$

Utility of the theorem on the invariance under homotopy of the line integral of a local gradient. Theorem 8.20 is a key step in proving existence results for primitives on a simply connected open set using the *gluing theorem for local primitives* (Theorem 9.4):
— Primitive of a field of $\mathcal{C}^1$ functions satisfying Poincaré's condition (Theorem 9.10) or of a merely continuous field satisfying a weaker version of this condition (Theorem 9.11).
— Stream function of a two-dimensional divergence-free field (Theorem 9.12). □

Proof of Theorem 8.20. *Intermediate closed paths.* After reparametrizing Γ and Γ_* with Theorem 8.9 if necessary, we may assume that they are defined on $[0, 1]$. Let H be a homotopy between Γ and Γ_* in Ω; in other words, let $H \in \mathcal{C}([0, 1] \times [0, 1]; \Omega)$ such that, for every t and s in $[0, 1]$,

$$H(t, 0) = \Gamma(t), \quad H(t, 1) = \Gamma_*(t), \quad H(0, s) = H(1, s).$$

Define $N + 1$ closed paths $\Gamma_n \in \mathcal{C}([0, 1]; \Omega)$, where $n \in [\![0, N]\!]$, by

$$\Gamma_n(t) = H\Big(t, \frac{n}{N}\Big).$$

Split each Γ_n into N pieces $\Gamma_n^m \in \mathcal{C}([m/N, (m+1)/N]; \Omega)$, where $m \in [\![0, N-1]\!]$, defined by $\Gamma_n(t) = H(t, n/N)$, so (see Figure 8.2),

$$\Gamma_n = \Gamma_n^0 \overrightarrow{\cup} \Gamma_n^1 \overrightarrow{\cup} \dots \overrightarrow{\cup} \Gamma_n^{N-1}.$$

Finally, define the intermediate points a_n^m, where $n \in [\![0, N]\!]$ and $m \in [\![0, N]\!]$, by

$$a_n^m = H\Big(\frac{m}{N}, \frac{n}{N}\Big)$$

and denote by $T_n^m = \Gamma_{\overrightarrow{a_n^m, a_{n+1}^m}}$ the *transverse* rectilinear path joining a_n^m to a_{n+1}^m.

forms taking values in a Banach space. It can be seen, for example, in [CARTAN, Henri, 19, Theorem 3.7.3, p. 229], where the field q is "hidden" behind the 1-form ω and the existence of f_B such that $\nabla f_B = q$ corresponds to the hypothesis that "ω is closed".

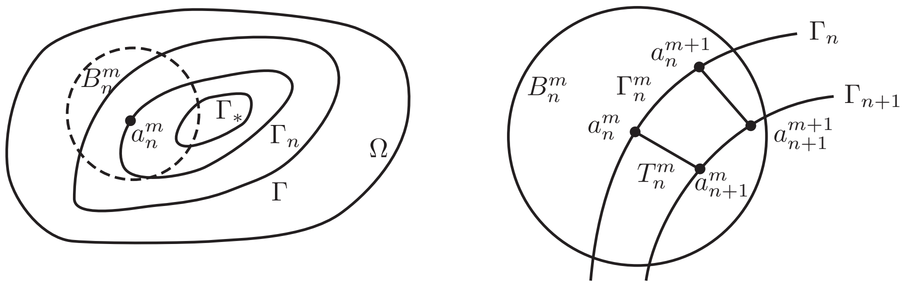

Figure 8.2. *Intermediate closed paths*

The image $[H] = \{H(t,s) : 0 \le t \le 1,\ 0 \le s \le 1\}$ is compact as the image of a compact set, here $[0,1] \times [0,1]$, under a continuous mapping (Theorem A.33). Thus, by the strong inclusion theorem (Theorem A.22), there exists $\delta > 0$ such that

$$[H] + B(0,\delta) \subset \Omega.$$

Choose N sufficiently large that $|t - t'| \le 1/N$ and $|s - s'| \le 1/N$ imply that $|H(t,s) - H(t',s')| \le \delta/3$, and let B_n^m be the open ball of center a_n^m and radius $2\delta/3$. Then

the paths Γ_n^m, Γ_{n+1}^m, T_n^m and T_n^{m+1} are included in B_n^m.

Invariance of the line integral along the Γ_n. By the hypotheses, there exists a function $f_n^m \in \mathcal{C}^1(B_n^m; E)$ such that

$$q = \nabla f_n^m \text{ on } B_n^m.$$

The formula for the line integral of a gradient (Theorem 8.11 (a)) gives

$$\begin{aligned}
\int_{\Gamma_{n+1}^m} q \cdot \mathrm{d}\ell - \int_{\Gamma_n^m} q \cdot \mathrm{d}\ell &= \int_{\Gamma_{n+1}^m} \nabla f_n^m \cdot \mathrm{d}\ell - \int_{\Gamma_n^m} \nabla f_n^m \cdot \mathrm{d}\ell = \\
&= \big(f_n^m(a_{n+1}^{m+1}) - f_n^m(a_{n+1}^m)\big) - \big(f_n^m(a_n^{m+1}) - f_n^m(a_n^m)\big) = \\
&= \int_{T_n^{m+1}} q \cdot \mathrm{d}\ell - \int_{T_n^m} q \cdot \mathrm{d}\ell.
\end{aligned}$$

Summing over m from 0 to $N-1$, we obtain

$$\int_{\Gamma_{n+1}} q \cdot \mathrm{d}\ell - \int_{\Gamma_n} q \cdot \mathrm{d}\ell = \int_{T_n^N} q \cdot \mathrm{d}\ell - \int_{T_n^0} q \cdot \mathrm{d}\ell.$$

The right-hand side of the equation is zero because the paths T_n^0 and T_n^N coincide. Indeed, T_n^0 joins the initial points a_n^0 and a_{n+1}^0 of Γ_n and Γ_{n+1}, while T_n^N joins their ending points a_n^N and a_{n+1}^N, and these ending points coincide with the initial points, since

$$a_n^0 = H\Big(0, \frac{n}{N}\Big) = H\Big(1, \frac{n}{N}\Big) = a_n^N,$$

and similarly $a_{n+1}^0 = a_{n+1}^N$. Therefore,

$$\fint_{\Gamma_{n+1}} q \cdot \mathrm{d}\ell - \fint_{\Gamma_n} q \cdot \mathrm{d}\ell = 0_E.$$

This holds for each n, so

$$\fint_{\Gamma_N} q \cdot \mathrm{d}\ell = \fint_{\Gamma_0} q \cdot \mathrm{d}\ell.$$

This proves the stated result, since $\Gamma_0 = \Gamma$ and $\Gamma_N = \Gamma_*$. $\square$

Stokes' formula. Theorem 8.20 on the invariance under homotopy is a (non-elementary) variant of Stokes' formula[6]

$$\int_{\partial H} \sigma = \int_H \mathrm{d}\sigma,$$

where σ is an exterior differential k-form and H is a $k{+}1$-chain. Details may be found, for example, in [Bourbaki, 15, § 11.3.4, p. 49], when σ takes values in a Banach space E.

Indeed, the hypothesis $\nabla f_B = q$ gives $\partial_i q_j = \partial_i \partial_j f_B = \partial_j \partial_i f_B = \partial_j q$, so the differential 1-form $\sigma = \Sigma_j q_j \, \mathrm{d}x_j$ satisfies

$$\mathrm{d}\sigma = \Sigma_{ij} \partial_i q_j \, \mathrm{d}x_i \wedge \mathrm{d}x_j = \Sigma_{i<j} (\partial_i q_j - \partial_j q_i) \, \mathrm{d}x_i \wedge \mathrm{d}x_j = 0_E.$$

Since a homotopy H between Γ and Γ_* is an oriented 2-chain with boundary $\partial H = \overrightarrow{\Gamma} \cup \overleftarrow{\Gamma_*}$, it follows that

$$\fint_{\Gamma} q \cdot \mathrm{d}\ell - \fint_{\Gamma_*} q \cdot \mathrm{d}\ell = \fint_{\partial H} \sigma = \fint_H \mathrm{d}\sigma = 0_E. \ \square$$

6. **History of Stokes' formula.** We did not find any specific references about the origin of this formula. Attributed to Sir George Gabriel Stokes, it was supposedly discovered by Mikhail Vasilyevitch Ostrogradsky around 1820 and then rediscovered by Lord Kelvin. It can also be found associated with names such as Carl Friedrich Gauss and George Green in various forms, among which is the formula of Theorem 10.8.

Chapter 9

Primitives of Continuous Functions

The purpose of this chapter is determine the conditions under which a continuous field $q = (q_1, \dots, q_d)$ admits a primitive f, that is, a function f satisfying $\nabla f = q$.

We begin by explicitly constructing a primitive when $\oint_\Gamma q \cdot \mathrm{d}\ell = 0$ for every closed path Γ in Ω (Theorem 9.1) by integrating q along paths. From this, we deduce that it is sufficient for q to be orthogonal to the divergence-free test fields, in other words, to have $\int_\Omega q \cdot \psi = 0$ for every ψ such that $\nabla \cdot \psi = 0$ (Theorem 9.2), using a tubular flow as a test field and its integral concentration property. This is the *orthogonality theorem*. These conditions are in fact both necessary and sufficient.

We then show that, when Ω is simply connected, it suffices that there exists a primitive on every ball $B \subset \Omega$ (Theorem 9.4), thanks to the theorem on the invariance under homotopy of the line integral. This is the *primitive gluing theorem*. We therefore construct such local primitives:
— When q is $\mathcal{C}^1$ and $\partial_i q_j = \partial_j q_i$ (for every i and j), by integrating q along line segments (Theorem 9.5). This is *Poincaré's theorem*.
— When q is merely continuous and $\int_\Omega q_j \partial_i \varphi = \int_\Omega q_i \partial_j \varphi$ for every test function φ (Theorem 9.7), by regularization. This condition is a weak version of the Poincaré condition.

Thus, when Ω is simply connected, there exists a primitive whenever either the Poincaré condition or its weak version are satisfied (Theorems 9.10 and 9.11). These conditions are necessary and sufficient.

We compare these conditions in Theorem 9.14. Finally, we show that, if the primitive exists, it is unique up to an additive constant on each connected component Ω_m of Ω and that, by fixing its values at a point of each Ω_m, we obtain a continuous mapping $q \mapsto f$ (Theorem 9.18).

9.1. Explicit primitive of a field with line integral zero

Let us explicitly construct a **primitive** q^* of a field $q = (q_1, \dots, q_d)$, that is, a function q^* satisfying $\nabla q^* = q$, whenever the line integral of q is zero around every closed path[1].

1. **History of the explicit construction of a primitive in Theorem 9.1.** We do not know the origin of this result, which is a classical result of the theory of differential forms taking values in a Banach space (see, for example, [Henri CARTAN, 19, Theorem 3.4.3, p. 220], where q is "hidden" behind the differential form ω).

THEOREM 9.1.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space, such that, for every closed path Γ in Ω of class $\mathcal{C}^1$,*

$$\oint_\Gamma q \cdot \mathrm{d}\ell = 0_E. \tag{9.1}$$

On each connected component Ω_m of Ω, choose a point a_m. Then:

(a) *For each m and each $x \in \Omega_m$, the element of E defined by*

$$q^*(x) \stackrel{\text{def}}{=} \int_{\Gamma(a_m, x)} q \cdot \mathrm{d}\ell$$

is independent of the path $\Gamma(a_m, x)$ of class $\mathcal{C}^1$ joining a_m to x in Ω_m (such a path always exists).

(b) *We have $q^* \in \mathcal{C}^1(\Omega; E)$ and*

$$\nabla q^* = q.$$

(c) *If the line segment $[a_m, x]$ is included in Ω,*

$$q^*(x) = (x - a_m) \cdot \int_0^1 q(a_m + t(x - a_m))\, \mathrm{d}t. \ \blacksquare$$

Optimality of Theorem 9.1 (b). The condition (9.1) is necessary and sufficient for q to have a primitive because, if $q = \nabla q^*$, then $\oint_\Gamma q \cdot \mathrm{d}\ell = 0$, since the line integral of a gradient around a closed path is always zero (Theorem 8.11 (b)). $\square$

Inconsistent notation. Here, we have denoted the primitive by q^*, but elsewhere it is always denoted by f. This is intentional to highlight that q^* is a special, explicit primitive, whereas f is arbitrary. $\square$

Proof of Theorem 9.1. (a) Since each connected component Ω_m of Ω is connected and open (Theorem A.16), each of its points x is joined (Theorem 8.5) to the point a_m by a path $\Gamma(a_m, x)$ in Ω_m, and therefore in Ω, of class $\mathcal{C}^1$.

Let us check that $\int_{\Gamma(a_m,x)} q \cdot \mathrm{d}\ell$ does not depend on the path Γ joining a_m to x. Let Γ and Γ_* be two such paths. The concatenation $\Gamma \vec{\cup} \overleftarrow{\Gamma_*}$ of Γ and of the reverse path of Γ_* is a piecewise closed $\mathcal{C}^1$ path. By Definition 8.15 of the line integral along a concatenation, and since the sign of the line integral changes along the reverse path by Theorem 8.8,

$$\int_{\Gamma \vec{\cup} \overleftarrow{\Gamma_*}} q \cdot \mathrm{d}\ell = \int_\Gamma q \cdot \mathrm{d}\ell + \int_{\overleftarrow{\Gamma_*}} q \cdot \mathrm{d}\ell = \int_\Gamma q \cdot \mathrm{d}\ell - \int_{\Gamma_*} q \cdot \mathrm{d}\ell.$$

Reparameterizing this path using Theorem 8.4, we obtain, by Theorem 8.16, a closed $\mathcal{C}^1$ path with the same line integral, which is zero by the hypotheses. Therefore, we indeed have

$$\fint_{\Gamma} q \cdot \mathrm{d}\ell = \fint_{\Gamma_*} q \cdot \mathrm{d}\ell.$$

(b) We need to show that q^* is continuously differentiable and $\partial_i q^* = q_i$. Let $x \in \Omega$, $\eta > 0$ such that the ball $\{y \in \mathbb{R}^d : |y - x| \leq \eta\}$ is included in Ω, and let s be a non-zero real number such that $|s| \leq \eta$.

Again by Definition 8.15 and Theorem 8.8, the definition of q^* gives

$$q^*(x + s\mathbf{e}_i) - q^*(x) = \fint_{\Gamma(a, x+s\mathbf{e}_i)} q \cdot \mathrm{d}\ell - \fint_{\Gamma(a,x)} q \cdot \mathrm{d}\ell = \fint_{\Lambda} q \cdot \mathrm{d}\ell,$$

where $\Lambda = \overleftarrow{\Gamma(a,x)} \,\vec{\cup}\, \Gamma(a, x + s\mathbf{e}_i)$. Since the path Λ joins x to $x + s\mathbf{e}_i$ and the line integral is independent of the path joining these two points by (a), this equation holds when Λ is chosen to be the rectilinear path $\Gamma_{\overrightarrow{x, x+s\mathbf{e}_i}}$. The formula for the line integral along such a path from Theorem 8.10 (b) gives

$$q^*(x + s\mathbf{e}_i) - q^*(x) = s\mathbf{e}_i \cdot \int_0^1 q(x + ts\mathbf{e}_i)\,\mathrm{d}t = s \int_0^1 q_i(x + ts\mathbf{e}_i)\,\mathrm{d}t.$$

Therefore,

$$q^*(x + s\mathbf{e}_i) - q^*(x) - s q_i(x) = s \int_0^1 (q_i(x + ts\mathbf{e}_i) - q_i(x))\,\mathrm{d}t.$$

For every semi-norm $\| \; \|_{E;\nu}$ of E, the bound on the semi-norms of the Cauchy integral from Theorem 4.15 gives

$$\|q^*(x + s\mathbf{e}_i) - q^*(x) - s q_i(x)\|_{E;\nu} \leq |s| \sup_{0 \leq r \leq s} \|q_i(x + r\mathbf{e}_i) - q_i(x)\|_{E;\nu}.$$

For every $\epsilon > 0$, we can choose η such that the right-hand side is $\leq \epsilon\,|s|$ because q_i is continuous, so by the characterization of the partial derivatives (2.7) from Definition 2.8,

$$\partial_i q^*(x) = q_i(x).$$

Since its partial derivatives are continuous, q^* indeed belongs to $\mathcal{C}^1(\Omega; E)$ by Theorem 2.10.

(c) This is again the formula for the line integral along a rectilinear path from Theorem 8.10 (b). □

Connected components of an open subset of $\mathbb{R}^d$. Let us observe that the number of points a_m that must be chosen in Theorem 9.1 is countable (and possibly finite) since:

Every open subset U of $\mathbb{R}^d$ has, at most, countably many connected components. (9.2)

Proof. By Theorem A.16, the connected components of U are pairwise disjoint and each of them is open and therefore contains a point of $\mathbb{Q}^d$. Therefore, the set of them is countable as the image of a countable set (Theorem A.2 (b)), here a subset of $\mathbb{Q}^d$ (which is countable by Theorem A.2 (a), (d) and (c)). $\square$

9.2. Primitive of a field orthogonal to the divergence-free test fields

Let us show that a field $q = (q_1, \ldots, q_d)$ has a primitive f whenever it is “orthogonal” to every divergence-free **test field** $\psi = (\psi_1, \ldots, \psi_d)$. This is the **orthogonality theorem**[2].

THEOREM 9.2.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space, such that:*

$$\int_\Omega q \cdot \psi = 0_E, \quad \forall \psi \in \mathcal{K}^\infty(\Omega; \mathbb{R}^d) \text{ such that } \nabla \cdot \psi = 0. \tag{9.3}$$

Then there exists $f \in \mathcal{C}^1(\Omega; E)$ such that

$$\nabla f = q. \ \blacksquare$$

Optimality of Theorem 9.2. The condition (9.3) is necessary and sufficient for q to have a primitive, since, if $q = \nabla f$, then $\nabla \cdot \psi = 0$ implies

$$\int_\Omega q \cdot \psi = \int_\Omega \sum_{i=1}^d \partial_i f \, \psi_i = -\int_\Omega f \sum_{i=1}^d \partial_i \psi_i = -\int_\Omega f \, \nabla \cdot \psi = 0_E. \ \square$$

2. **History of the existence of primitives for fields that are orthogonal to the divergence-free test fields. Real values.** Theorem 9.2 is a special case of the *orthogonality theorem for distributions* given in Vol. 3, which follows from the *cohomology theorem* of Georges DE RHAM in the case $E = \mathbb{R}$. The latter showed in 1955 [28, Theorem 17’, p. 114] that a *current* T is homologous to 0 if and only if $T(\psi) = 0$ for every form ψ that is $\mathcal{C}^\infty$, closed, and has compact support (currents generalize differential forms on a manifold in the same way that distributions generalize functions; for a differential form, this result means that *every closed differential form is exact*).
Jacques-Louis LIONS observed in 1969 [56, p. 69] that the orthogonality theorem for real distributions, and therefore for continuous functions, follows from the result of DE RHAM by considering the current $T = q_1 \mathrm{d}x_1 + \cdots + q_n \mathrm{d}x_n$ (a good explanation of the transition from differential forms to primitives is given for functions in [RUDIN, 66, § 10.42 and 10.43, pp. 262–264]).
In light of the importance of this result when solving the Navier–Stokes equations, various more direct and elementary proofs have been given for specific examples of distributions or real functions: by Olga LADYZHENSKAYA in 1963 [50, Theorem 1, p. 28] for $q \in (L^2(\Omega))^d$; by Luc TARTAR in 1978 [78] for $q \in (H^{-1}(\Omega))^d$; and by Jacques SIMON in 1993 [70] for every $q \in (\mathcal{D}'(\Omega))^d$.

Vector values. Jacques SIMON proved Theorem 9.2 for a Banach space E in 1993 [71, Theorem 5 (i), p. 4]. Here, we use the same method, which is based on the concentration theorem (Theorem 8.18). (The proof given by Georges DE RHAM [28] does not seem to extend to this case, since it uses the reflexivity properties of spaces of currents.)

Orthogonality. Generalizing the notion of orthogonality with respect to a scalar product, we can say that a field q satisfying (9.3) is *orthogonal to* $\mathcal{K}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^d)=\{\psi\in\mathcal{K}^\infty(\Omega;\mathbb{R}^d):\nabla\cdot\psi=0\}$ with respect to the bilinear mapping $(q,\psi)\mapsto\int_\Omega q\cdot\psi$ from $\mathcal{C}(\Omega;E^d)\times\mathcal{K}^\infty(\Omega;\mathbb{R}^d)$ into E.

We can say that the space $\mathcal{C}_\nabla(\Omega;E^d)$ of fields that are gradients is *the orthogonal complement of the space* $\mathcal{K}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^d)$ since the condition (9.3) is equivalent to $q\in\mathcal{C}_\nabla(\Omega;E^d)$. This can be denoted as

$$\mathcal{C}_\nabla(\Omega;E^d)=(\mathcal{K}^\infty_{\mathrm{div}}(\Omega;\mathbb{R}^d))^\perp.\ \square$$

Proof of Theorem 9.2. Let Γ be a closed path $\mathcal{C}^1$ in Ω. Since its image $[\Gamma]$ is compact, by the separation theorem (Theorem A.22), there exists $r>0$ such that the tube $\mathcal{T}=[\Gamma]+B(0,r)$ is included in Ω. Let n_Γ be an integer $\geq 1/r$.

For every $n\geq n_\Gamma$, let $\Psi_n\in\mathcal{C}^\infty_{\mathcal{T}_n}(\mathbb{R}^d;\mathbb{R}^d)$, where $\mathcal{T}_n=[\Gamma]+B(0,1/n)$, be the tubular flow given by Theorem 8.17, related to the regularizing function ρ_n given by Definition 7.7 (a). It satisfies $\nabla\cdot\Psi_n=0$ and its restriction belongs to $\mathcal{K}^\infty(\Omega;\mathbb{R}^d)$ by Theorem 2.16 (c), so the hypothesis (9.3) gives

$$\int_\Omega q\cdot\Psi_n=0_E.$$

The concentration theorem (Theorem 8.18) then gives

$$\int_\Gamma q\diamond\rho_n\cdot\mathrm{d}\ell=\int_\Omega q\cdot\Psi_n=0_E. \tag{9.4}$$

Now, $[\Gamma]\subset\Omega_{B(0,r)}\subset\Omega_{B(0,1/n_\Gamma)}$ by Theorem 7.3 and therefore $q\diamond\rho_n\to q$ in $\mathcal{C}(\Omega_{B(0,1/n_\Gamma)};E)$ by Theorem 7.9 (a). Therefore,

$$\int_\Gamma q\diamond\rho_n\cdot\mathrm{d}\ell\to\int_\Gamma q\cdot\mathrm{d}\ell,$$

since the line integral depends continuously on q (Theorem 8.12 (b)) and hence sequentially continuously on q (Theorem A.29). In the limit,

$$\int_\Gamma q\cdot\mathrm{d}\ell=0_E.$$

This implies the existence of f such that $\nabla f=q$ by Theorem 9.1. $\square$

9.3. Gluing of local primitives on a simply connected open set

Let us define the notion of simply connected set.

DEFINITION 9.3.– *A subset U of a separated semi-normed space is said to be* ***simply connected*** *if every closed path of U is homotopic in U to a closed path consisting of a single point.* ▮

Simple connectedness versus connectedness. The term "simply connected" is unfortunately not defined in the same way by all authors. For some, it requires connectedness, which is not the case here.

To require connectedness, it suffices to replace the condition "every closed path is homotopic to **a** point" with "every closed path is homotopic to **every** point of U." □

Simple connectedness in $\mathbb{R}^d$ versus the presence of "holes":
— The space $\mathbb{R}^d$ is simply connected (this can be verified by choosing $T(t,s) = (1-s)\,\Gamma(t)$).
— In $\mathbb{R}$, every open set is simply connected, even if it has holes.
— In $\mathbb{R}^2$, an open set is simply connected if and only if it does not have holes. For instance, the *crown* $\{x \in \mathbb{R}^2 : 1 < |x| < 2\}$ is connected but not simply connected.
— In $\mathbb{R}^d$, $d \geq 3$, simply connected open sets can have holes. For instance, the set $\{x \in \mathbb{R}^3 : 1 < |x| < 2\}$ is both connected and simply connected. □

Simple connectedness of star-shaped sets. In a separated semi-normed space:

$$\text{Every star-shaped set is connected and simply connected.} \tag{9.5}$$

Proof. Let U be a set that is star shaped with respect to a point a, that is, for every $z \in U$, the line segment $[a,z]$ is included in U.

It is simply connected because every closed path Γ is homotopic in U to the path consisting of the single point $\{a\}$ via the homotopy $H(t,s) = sa + (1-s)\Gamma(t)$.

It is connected because if it was covered by two disjoint non-empty open sets $\mathcal{O}_1$ and $\mathcal{O}_2$, then a would belong to one of them, say $\mathcal{O}_1$, and $\mathcal{O}_2$ would contain a point z of U. The sets $\mathcal{U}_i = \{s \in \mathbb{R} : a + s(z-a) \in \mathcal{O}_i\}$ would then form an open covering of the interval $[0,1]$, which contradicts its connectedness (Theorem A.16). □

Let us show that, on a simply connected open set, if a field is locally a gradient, then it is a gradient. In other words, if the field has local primitives, then it has a global primitive. We call this the **local primitive gluing theorem**[3].

THEOREM 9.4.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where E is a Neumann space and*

$$\Omega \text{ is a simply connected open subset of } \mathbb{R}^d,$$

such that, for every open ball $B \Subset \Omega$, there exists $f_B \in \mathcal{C}^1(B; E)$ such that

$$\nabla f_B = q \text{ on } B.$$

Then there exists $f \in \mathcal{C}^1(\Omega; E)$ such that

$$\nabla f = q. \quad ▮$$

3. **History of the local primitive gluing theorem.** We are not familiar with the origin of Theorem 9.4, which is a classical result from the theory of differential forms taking values in a Banach space. It can be seen, for example, in [Henri CARTAN, 19, Theorem 3.8.1, p. 230], where q is "hidden" behind the *closed differential* 1*-form* ω, where *closed* means that it is *locally a gradient* [19, Definition, p. 222].

Proof. Let Γ be a closed path in Ω of class $\mathcal{C}^1$. By Definition 9.3 of a simply connected open set, Γ is homotopic in Ω to a closed path Γ_* consisting of a single point. Since q is locally a gradient by the hypotheses, its line integral around a closed path is invariant under homotopy by Theorem 8.20, so

$$\oint_\Gamma q \cdot \mathrm{d}\ell = \oint_{\Gamma_*} q \cdot \mathrm{d}\ell.$$

Since the line integral around a closed path consisting of a single point is zero (Theorem 8.10 (a)),

$$\oint_\Gamma q \cdot \mathrm{d}\ell = 0_E.$$

This holds for every Γ, which guarantees the existence of a primitive f by Theorem 9.1. □

Optimality of Theorem 9.4. The existence of local primitives is clearly a necessary condition for the existence of a global primitive. This condition is therefore necessary and sufficient.

On an arbitrary open set, the result no longer holds in general. An example of a function with local primitives but no global primitive is given in Theorem 9.15, in two dimensions, and in Theorem 9.16, in arbitrary dimensions. □

Caution. Theorem 9.4 is a *gluing theorem for* **certain** *local primitives*, but **not all** local primitives. Indeed, *it may be impossible to glue together the given local primitives* f_B because they may differ by a constant. However, when Ω is simply connected, we can add a constant to each of them to glue them together. □

9.4. Explicit primitive on a star-shaped set: Poincaré's theorem

A subset U of a vector space is said to be **star shaped with respect to the point** a if, for every $u \in U$, it contains the line segment $[a, u] = \{a + t(u - a) : 0 \le t \le 1\}$. A set is said to be **star shaped** if it is star shaped with respect to one of its points.

Let us explicitly construct a primitive of a continuously differentiable vector field $q = (q_1, \ldots, q_d)$ such that $\partial_i q_j = \partial_j q_i$ for every i and j on a star-shaped open set, a result that is weaker (see Theorem 9.14 (e)) than the conditions (9.1) and (9.3) considered earlier for an arbitrary open set. This is called **Poincaré's theorem**[4].

THEOREM 9.5.– *Let $q \in \mathcal{C}^1(\Omega; E^d)$, where E is a Neumann space and*

Ω is an open subset of $\mathbb{R}^d$ that is star shaped with respect to a point a,

4. **History of Poincaré's theorem.** Theorem 9.5 was established by Henri POINCARÉ in 1899 [63, p. 10] in the real-valued case.

such that, for every i and j in $[\![1, d]\!]$,

$$\partial_i q_j = \partial_j q_i.$$

Then the function defined, for every $x \in \Omega$, by

$$q^*(x) \stackrel{\text{def}}{=} (x - a) \cdot \int_0^1 q(a + t(x - a))\, \mathrm{d}t$$

satisfies $q^* \in \mathcal{C}^2(\Omega; E)$ and

$$\nabla q^* = q. \quad ▮$$

Proof. For every $x \in \Omega$ and $j \in [\![1, d]\!]$, we denote

$$Q_j(x) = \int_0^1 q_j(a + t(x - a))\, \mathrm{d}t.$$

Suppose for now that differentiation under the integral sign is admissible, so that $Q_j \in \mathcal{C}^1(\Omega; E)$ and

$$\partial_i Q_j(x) = \int_0^1 t\, \partial_i q_j(a + t(x - a))\, \mathrm{d}t. \tag{9.6}$$

Then $q^*(x) = \sum_{j=1}^d (x_j - a_j) Q_j(x)$, so Theorem 3.6 and the Leibniz rule give $q^* \in \mathcal{C}^1(\Omega; E)$ and

$$\begin{aligned}\partial_i q^*(x) = Q_i(x) + \sum_{j=1}^d (x_j - a_j)\partial_i Q_j(x) = \\ = \int_0^1 q_i(a + t(x - a)) + t \sum_{j=1}^d (x - a)_j \partial_i q_j(a + t(x - a))\, \mathrm{d}t.\end{aligned}$$

Now, apply the hypothesis $\partial_i q_j = \partial_j q_i$ and observe that, by Theorem 3.12 (a) on changes of variables in a derivative with $T(t) = a + t(x - a)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(q_i(a + t(x - a))\big) = \sum_{j=1}^d \partial_j q_i(a + t(x - a))\,(x - a)_j,$$

since $\mathrm{d}T_j/\mathrm{d}t(t) = (x - a)_j$. We therefore obtain

$$\partial_i q^*(x) = \int_0^1 q_i(a + t(x - a)) + t\, \frac{\mathrm{d}}{\mathrm{d}t}\big(q_i(a + t(x - a))\big)\, \mathrm{d}t.$$

By the Leibniz rule once again, together with the expression of the integral of a derivative from Theorem 6.4 (a), we finally obtain

$$\partial_i q^*(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(t\, q_i(a + t(x-a))\big)\,\mathrm{d}t = q_i(x).$$

This proves that

$$\nabla q^* = q.$$

Hence, $\partial_j \partial_i q^* = \partial_j q_i \in \mathcal{C}(\Omega; E)$, and so $q^* \in \mathcal{C}^2(\Omega; E)$.

We still need to check (9.6). Let B be an open ball such that $\overline{B} \subset \Omega$. Theorem 4.27 on differentiating under the integral sign on $B \times (0,1)$ with $f(x,t) = q_j(a+t(x-a))$ and $g_i(x,t) = t\,\partial_i q_j(a + t(x-a))$ then gives (9.6) on each B and therefore on the whole of Ω. Indeed, the hypotheses of this theorem are satisfied:
— The functions f and g_i are uniformly continuous and bounded because they are uniformly continuous and bounded on the compact set $\overline{B} \times [0,1]$ by Heine's theorem (Theorem A.34), since they are continuous on this set.
— For any fixed t, the mapping $x \mapsto f(x,t)$ is differentiable and $\partial_i f(x,t) = g_i(x,t)$. This is an elementary fact, which completes the proof of (9.6), and hence the proof of Theorem 9.5. □

9.5. Explicit primitive under the weak Poincaré condition

Before we consider the weak Poincaré condition, let us observe that every open set Ω that is star shaped, with respect to a point a, is the union of the subsets $\Omega^{*a}_{1/n}$ of the $\Omega_{1/n}$ that are star shaped with respect to a.

THEOREM 9.6.– *Let Ω be an open subset of $\mathbb{R}^d$ that is star shaped with respect to a point a and, for every $n \in \mathbb{N}$, let*

$$\Omega^{*a}_{1/n} \overset{\text{def}}{=} \{x \in \Omega : [a,x] \subset \Omega_{1/n}\},$$

where $[a,x] = \{a + t(x-a) : 0 \le t \le 1\}$ and $\Omega_{1/n} = \{x \in \Omega : B(x, 1/n) \subset \Omega\}$.

Then $\Omega^{*a}_{1/n}$ is a star shaped open set and

$$\Omega = \bigcup_{n \in \mathbb{N}} \Omega^{*a}_{1/n}. \blacksquare$$

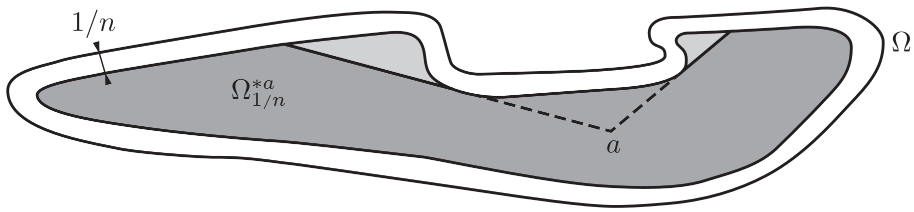

Figure 9.1. *Subset $\Omega^{*a}_{1/n}$ of $\Omega_{1/n}$ that is star shaped with respect to a. $\Omega^{*a}_{1/n}$ is dark gray and $\Omega_{1/n}$ is the union of the light gray and dark gray regions*

Proof of Theorem 9.6. The set $\Omega^{*a}_{1/n}$ is star shaped because, if $x \in \Omega^{*a}_{1/n}$, then, for all $y \in [a, x]$, we have $[a, y] \subset [a, x] \subset \Omega_{1/n}$, so $y \in \Omega^{*a}_{1/n}$ and hence $[a, x] \subset \Omega^{*a}_{1/n}$.

Let us show that this set is open. Let $x \in \Omega^{*a}_{1/n}$. Then $[a, x]$ is a compact set included in $\Omega_{1/n}$, which is open (Theorem 7.2 (a)), so the strong inclusion theorem (Theorem A.22) gives $r > 0$ such that $[a, x] + B(0, r) \subset \Omega_{1/n}$. If $y \in B(x, r)$,

$$[a, y] = \{a + t(x - a) + t(y - x) : 0 \le t \le 1\} \subset [a, x] + B(0, r) \subset \Omega_{1/n},$$

since $|t(y - x)| \le tr \le r$, so $y \in \Omega^{*a}_{1/n}$. This proves that $\Omega^{*a}_{1/n}$ is open.

Finally, Ω is the union of the $\Omega^{*a}_{1/n}$ because, if $x \in \Omega$, then $[a, x] \subset \Omega$ so, again by Theorem A.22, there exists $r > 0$ such that $[a, x] + B(0, r) \subset \Omega$; then $[a, x] \subset \Omega_{1/n}$, that is, $x \in \Omega^{*a}_{1/n}$, whenever $n \ge 1/r$. $\square$

If q is merely continuous, Poincaré's condition $\partial_i q_j = \partial_j q_i$ is no longer meaningful in the "classical" sense, but we can state a "weak" formulation that still guarantees the existence of an explicit primitive on a star-shaped open set as follows:

THEOREM 9.7.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where E is a Neumann space and*

Ω is an open subset of $\mathbb{R}^d$ that is star shaped with respect to a point a,

such that, for every i and j in $[\![1, d]\!]$ and every $\varphi \in \mathcal{K}^\infty(\Omega)$,

$$\int_\Omega q_j \,\partial_i \varphi = \int_\Omega q_i \,\partial_j \varphi. \tag{9.7}$$

Then the function defined by

$$q^*(x) \stackrel{\text{def}}{=} (x - a) \cdot \int_0^1 q(a + t(x - a))\,\mathrm{d}t$$

satisfies $q^* \in \mathcal{C}^1(\Omega; E)$ and

$$\nabla q^* = q. \ \blacksquare$$

Weak Poincaré condition. We name *weak Poincaré condition* the equality (9.7) because, when $q \in \mathcal{C}^1(\Omega; E^d)$, it is equivalent to Poincaré's condition $\partial_i q_j = \partial_j q_i$ as we will see in Theorem 9.9. □

Proof of Theorem 9.7. We will proceed in four steps.

1. *Regularization.* Let $q \diamond \rho_n \in \mathcal{C}^\infty(\Omega_{1/n}; E^d)$ be a regularized function of q given by Definition 7.7, where the support of ρ_n is included in the ball $B(0, 1/n)$ and $\Omega_{1/n} = \Omega_{B(0,1/n)}$, namely the set $\{x \in \mathbb{R}^d : B(x, 1/n) \subset \Omega\}$. Let us show that, for every i and j,

$$\partial_i(q_j \diamond \rho_n) - \partial_j(q_i \diamond \rho_n) = 0_E \ \text{ on } \Omega_{1/n}. \tag{9.8}$$

The expression of the derivative of a weighting from Theorem 7.4 (b) and the second expression of the weighting itself from Theorem 7.2 (c) give

$$\partial_i(q \diamond \rho_n)(x) = -(q \diamond \partial_i \rho_n)(x) = -\int_\Omega q(y)\, \partial_i \rho_n(y - x)\, \mathrm{d}y.$$

Hence,

$$\partial_i(q_j \diamond \rho_n)(x) - \partial_j(q_i \diamond \rho_n)(x) = \int_\Omega q_i(y)\, \partial_j \rho_n(y - x) - q_j(y)\, \partial_i \rho_n(y - x)\, \mathrm{d}y.$$

The right-hand side is zero by the weak Poincaré condition (9.7) applied to φ defined by $\varphi(y) = \rho_n(y - x)$, which establishes (9.8).

2. *Primitive on star shaped subsets.* Let $\Omega^{*a}_{1/n}$ be the subset of $\Omega_{1/n}$ that is star shaped with respect to a. This set is open and star shaped (Theorem 9.6). By Poincaré's theorem (Theorem 9.5), the property (9.8) implies that the function defined on $\Omega^{*a}_{1/n}$ by

$$q^*_n(x) = (x - a) \boldsymbol{\cdot} \int_0^1 (q \diamond \rho_n)(a + t(x - a))\, \mathrm{d}t \tag{9.9}$$

is a primitive of $q \diamond \rho_n$. In other words, for every i,

$$\partial_i q^*_n = q_i \diamond \rho_n \ \text{ on } \Omega^{*a}_{1/n}.$$

3. *Convergence.* Let $k \in \mathbb{N}$ and $n \geq k$. Then $\Omega^{*a}_{1/k} \subset \Omega^{*a}_{1/n} \subset \Omega_{1/n}$ and the regularizing property from Theorem 7.10 (a) gives, as $n \to \infty$,

$$q_i \diamond \rho_n \to q_i \text{ on } \mathcal{C}(\Omega^{*a}_{1/k}; E).$$

That is,

$$\partial_i q^*_n \to q_i \ \text{ on } \mathcal{C}(\Omega^{*a}_{1/k}; E).$$

Moreover the expression (9.9) of q_n^* implies, as we will verify in Lemma 9.8,

$$q_n^* \to q^* \text{ on } \mathcal{C}(\Omega_{1/k}^{*a}; E). \tag{9.10}$$

The completeness property of $\mathcal{C}^1(\Omega_{1/k}^{*a}; E)$ from Theorem 2.23 then shows that $q^* \in \mathcal{C}^1(\Omega_{1/k}^{*a}; E)$ and

$$\partial_i q^* = q_i \text{ on } \Omega_{1/k}^{*a}.$$

4. *Gluing*. Since the $(\Omega_{1/k}^{*a})_{k \geq 1}$ cover Ω (Theorem 9.6), it follows that q^* belongs to $\mathcal{C}^1(\Omega; E)$ and $\nabla q^* = q$ on the whole of Ω. $\square$

We still need to establish the convergence (9.10), which corresponds to the following property.

LEMMA 9.8.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where Ω is a star-shaped open subset of $\mathbb{R}^d$ with respect to a point a and E is a Neumann space, and, for every $x \in \Omega$, let*

$$q^*(x) \stackrel{\text{def}}{=} (x-a) \cdot \int_0^1 q(a + t(x-a))\, \mathrm{d}t.$$

Then the mapping $q \mapsto q^*$ is linear and continuous, and therefore sequentially continuous, from $\mathcal{C}(\Omega; E^d)$ into $\mathcal{C}(\Omega; E)$. ▮

Proof. Let $\{\| \ \|_{E;\nu} : \nu \in \mathcal{N}_E\}$ be the family of semi-norms of E. Definition 1.3 (a) of the semi-norms of $\mathcal{C}(\Omega; E)$, together with the inequality (2.2) (p. 31) and the bound on the semi-norms of the integral from Theorem 4.17 (b), implies that, for every compact set $K \subset \Omega$ and every $\nu \in \mathcal{N}_E$,

$$\|q^*\|_{\mathcal{C}(\Omega;E);K,\nu} = \sup_{x \in K} \|q^*(x)\|_{E;\nu} \leq c \sup_{x \in D} \|q(x)\|_{E^d;\nu} = c\, \|q\|_{\mathcal{C}(\Omega;E^d);D,\nu},$$

where $c = \sup_{x \in K} |x - a|$ and $D = \{a + t(x-a) : x \in K,\ 0 \leq t \leq 1\}$. The final term is well defined (Definition 1.3 (a)) because D is compact in $\mathbb{R}^d$ (since it is closed and bounded) and included in Ω (since this set is star shaped).

By the characterization of continuous linear mappings from Theorem 1.25, this shows that the mapping $q \mapsto q^*$ is continuous. It is therefore sequentially continuous, like any continuous mapping (Theorem A.29). $\square$

Continuity taking values in $\mathcal{C}^1(\Omega; E)$. It follows from Lemma 9.8 and Theorem 9.7 that, if Ω is star shaped, the mapping $q \mapsto q^*$ is continuous from $\mathcal{C}(\Omega; E^d)$ into $\mathcal{C}^1(\Omega; E)$. Indeed, $q \mapsto \partial_i q^*$ is also continuous from $\mathcal{C}(\Omega; E^d)$ into $\mathcal{C}(\Omega; E)$ because $\partial_i q^* = q_i$.

This property is generalized to an arbitrary open set Ω in Theorem 9.18. $\square$

Let us check that the hypothesis (9.7) of Theorem 9.7 is a weak version of Poincaré's condition $\partial_i q_j = \partial_j q_i$.

THEOREM 9.9.– *Let $q \in \mathcal{C}^1(\Omega; E^d)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space. Then*

$$\partial_i q_j = \partial_j q_i,$$

where i and j belong to $[\![1, d]\!]$ if and only if, for every $\varphi \in \mathcal{K}^\infty(\Omega)$,

$$\int_\Omega q_j\, \partial_i \varphi = \int_\Omega q_i\, \partial_j \varphi. \blacksquare$$

Proof. *Direct implication.* If $\partial_i q_j = \partial_j q_i$, then, applying the formula of integration by parts from Theorem 6.12 twice, we obtain

$$\int_\Omega q_i\, \partial_j \varphi = -\int_\Omega \partial_j q_i\, \varphi = -\int_\Omega \partial_i q_j\, \varphi = \int_\Omega q_j\, \partial_i \varphi.$$

Converse. Let $\varphi \in \mathcal{K}^\infty(\Omega)$. Again applying the formula of integration by parts from Theorem 6.12, we obtain $\int_\Omega q_i\, \partial_j \varphi = -\int_\Omega \partial_j q_i\, \varphi$ and $\int_\Omega q_j\, \partial_i \varphi = -\int_\Omega \partial_i q_j\, \varphi$. Therefore, by subtraction,

$$\int_\Omega q_i\, \partial_j \varphi - q_j\, \partial_i \varphi = \int_\Omega (\partial_i q_j - \partial_j q_i)\, \varphi.$$

If this vanishes for every φ, then $\partial_i q_j = \partial_j q_i$ by the weak vanishing property from Theorem 6.13. □

9.6. Primitives on a simply connected open set

Let us show that, on a simply connected open set, Poincaré's condition $\partial_i q_j = \partial_j q_i$ guarantees the existence of a primitive for a continuously differentiable field.

THEOREM 9.10.– *Let $q \in \mathcal{C}^1(\Omega; E^d)$, where E is a Neumann space and*

Ω is a simply connected open subset of $\mathbb{R}^d$,

such that, for every i and j in $[\![1, d]\!]$,

$$\partial_i q_j = \partial_j q_i.$$

Then there exists $f \in \mathcal{C}^2(\Omega; E)$ such that

$$\nabla f = q. \blacksquare$$

Proof. By Poincaré's theorem (Theorem 9.5), the hypothesis $\partial_i q_j = \partial_j q_i$ implies the existence of a primitive on every ball $B \subset \Omega$. This implies the existence of a primitive on the whole of Ω by Theorem 9.4 on gluing together local primitives, since Ω is simply connected. □

Optimality of Theorem 9.10. The condition $\partial_i q_j = \partial_j q_i$ is necessary and sufficient for a continuously differentiable field q to have a primitive because, if $q = \nabla f$, then

$$\partial_i q_j = \partial_i \partial_j f = \partial_j \partial_i f = \partial_j q_i.$$

When Ω is simply connected, this condition is therefore necessary and sufficient.

For an arbitrary open set Ω, it is necessary but not always sufficient (Theorem 9.16). □

Let us remain in the case of a simply connected open set and show that, if q is merely continuous, the weak Poincaré condition also guarantees the existence of a primitive.

THEOREM 9.11.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where E is a Neumann space and*

$$\Omega \textit{ is a simply connected open subset of } \mathbb{R}^d,$$

such that, for every i and j in $[\![1, d]\!]$ and all $\varphi \in \mathcal{K}^\infty(\Omega)$,

$$\int_\Omega q_j\, \partial_i \varphi = \int_\Omega q_i\, \partial_j \varphi. \tag{9.11}$$

Then there exists $f \in \mathcal{C}^1(\Omega; E)$ such that

$$\nabla f = q. \ \blacksquare$$

Proof. By Theorem 9.7, the hypothesis (9.11) implies the existence of a primitive on every ball $B \subset \Omega$. This implies the existence of a primitive on the whole of Ω by Theorem 9.4 on gluing together local primitives, since Ω is simply connected. □

Optimality of Theorem 9.11. When Ω is simply connected, the condition (9.11) is necessary and sufficient for q to have a primitive because, if $q = \nabla f$, then, for every $\varphi \in \mathcal{K}^\infty(\Omega)$,

$$\int_\Omega q_j\, \partial_i \varphi = \int_\Omega \partial_j f\, \partial_i \varphi = -\int_\Omega f\, \partial_j \partial_i \varphi = -\int_\Omega f\, \partial_i \partial_j \varphi = \int_\Omega \partial_i f\, \partial_j \varphi = \int_\Omega q_i\, \partial_j \varphi.$$

For an arbitrary open set Ω, it is necessary but not always sufficient (Theorem 9.14 (e)). □

Let us show that, on a simply connected open subset of $\mathbb{R}^2$, every divergence-free field $v = (v_1, v_2)$ derives from a stream function. This is known as **Haar's lemma**[5].

THEOREM 9.12.– *Let $v \in \mathcal{C}^1(\Omega; E^2)$, where Ω is a simply connected open subset of $\mathbb{R}^2$ and E is a Neumann space, such that*

$$\partial_1 v_1 + \partial_2 v_2 = 0_E.$$

*Then there exists a **stream function** $f \in \mathcal{C}^2(\Omega; E)$ such that:*

$$v_1 = \partial_2 f, \quad v_2 = -\partial_1 f. \ \blacksquare$$

5. **History of Haar's lemma.** Alfréd HAAR showed Theorem 9.12 with $E = \mathbb{R}$ between 1926 [43] and 1929 [44].

Proof. The field $q = (-v_2, v_1)$ satisfies

$$\partial_1 q_2 - \partial_2 q_1 = \partial_1 v_1 + \partial_2 v_2 = 0_E,$$

so, by Theorem 9.10, it has a primitive f such that

$$\partial_1 f = q_1 = -v_2 \text{ and } \partial_2 f = q_2 = v_1. \ \square$$

Abbreviated formulation of Theorem 9.12. Denoting by $^\perp$ a rotation of $\pi/2$ in the counterclockwise direction, and so $\nabla^\perp = (\partial_2, -\partial_1)$, the result of Theorem 9.12 can be stated as follows:

If Ω is simply connected and $\nabla \cdot v = 0_E$, then there exists f such that $\nabla^\perp f = v$. (9.12)

$\square$

Uniqueness. The stream function f obtained in Theorem 9.12 is unique up to an additive constant on each connected component of Ω by Theorem 9.17 (b) (since ∇f is unique). $\square$

Weak existence condition. By Theorem 9.11, there exists a stream function $f \in \mathcal{C}^1(\Omega; E)$ whenever the divergence of the field $v = (v_1, v_2)$ is zero in the following weak sense: for every $\varphi \in \mathcal{K}^\infty(\Omega)$,

$$\int_\Omega v_1 \partial_1 \varphi + v_2 \partial_2 \varphi = 0_E. \ \square$$

Stream functions in arbitrary dimensions. In three and more dimensions, constructing a stream function associated with a divergence-free function becomes much more complex. Details may be found, for example, in [GIRAULT–RAVIART, 40, Chapter I, § 3.3]. $\square$

9.7. Comparison of the existence conditions for a primitive

Let us introduce the subspace of fields with a primitive.

DEFINITION 9.13.– *Let Ω be an open subset of $\mathbb{R}^d$ and E a Neumann space. We denote*

$$\mathcal{C}_\nabla(\Omega; E^d) \stackrel{\text{def}}{=} \{q \in \mathcal{C}(\Omega; E^d) : \exists f \in \mathcal{C}^1(\Omega; E) \text{ such that } \nabla f = q\},$$

which is a vector space that we endow with the semi-norms of $\mathcal{C}(\Omega; E^d)$. ▮

Let us compare the conditions used in previous sections to obtain the existence of a primitive.

THEOREM 9.14.– *Let $q \in \mathcal{C}(\Omega; E^d)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space. Then:*

(a) $q \in \mathcal{C}_\nabla(\Omega; E^d) \quad \Leftrightarrow \quad \exists f \in \mathcal{C}^1(\Omega; E)$ *such that* $\nabla f = q$

$$\Leftrightarrow \quad \int_\Omega q \cdot \psi = 0_E, \;\; \forall \psi \in \mathcal{K}^\infty(\Omega; \mathbb{R}^d) \textit{ such that } \nabla \cdot \psi = 0$$

$$\Leftrightarrow \quad \fint_\Gamma q \cdot \mathrm{d}\ell = 0_E, \;\; \forall \Gamma \textit{ closed } \mathcal{C}^1 \textit{ path in } \Omega.$$

(b) *If* Ω *is simply connected,*

$$q \in \mathcal{C}_\nabla(\Omega; E^d) \quad \Leftrightarrow \quad \exists f \in \mathcal{C}^1(\Omega; E) \textit{ such that } \nabla f = q$$

$$\Leftrightarrow \quad \int_\Omega q \cdot \psi = 0_E, \;\; \forall \psi \in \mathcal{K}^\infty(\Omega; \mathbb{R}^d) \textit{ such that } \nabla \cdot \psi = 0$$

$$\Leftrightarrow \quad \fint_\Gamma q \cdot \mathrm{d}\ell = 0_E, \;\; \forall \Gamma \textit{ closed } \mathcal{C}^1 \textit{ path in } \Omega$$

$$\Leftrightarrow \quad \fint_\Gamma q \cdot \mathrm{d}\ell = \fint_{\Gamma_*} q \cdot \mathrm{d}\ell, \;\; \forall \Gamma \textit{ and } \Gamma_* \textit{ homotopic closed } \mathcal{C}^1 \textit{ paths}$$

$$\Leftrightarrow \quad \int_\Omega q_j \partial_i \varphi = \int_\Omega q_i \partial_j \varphi, \;\; \forall i, \; \forall j, \; \forall \varphi \in \mathcal{K}^\infty(\Omega)$$

$$\Leftrightarrow \quad \forall \textit{ ball } B \Subset \Omega, \; \exists f_B \in \mathcal{C}^1(B; E) \textit{ such that } \nabla f_B = q \textit{ on } B.$$

(c) $\forall$ *ball* $B \Subset \Omega, \; \exists f_B \in \mathcal{C}^1(B; E)$ *such that* $\nabla f_B = q$ *on* B

$$\Leftrightarrow \quad \int_\Omega q_j \partial_i \varphi = \int_\Omega q_i \partial_j \varphi, \;\; \forall i, \; \forall j, \; \forall \varphi \in \mathcal{K}^\infty(\Omega)$$

$$\Leftrightarrow \quad \fint_\Gamma q \cdot \mathrm{d}\ell = \fint_{\Gamma_*} q \cdot \mathrm{d}\ell, \;\; \forall \Gamma \textit{ and } \Gamma_* \textit{ homotopic closed } \mathcal{C}^1 \textit{ paths.}$$

(d) *If* $q \in \mathcal{C}^1(\Omega; E^d)$*,*

$$\partial_i q_j = \partial_j q_i \quad \Leftrightarrow \quad \int_\Omega q_j \partial_i \varphi = \int_\Omega q_i \partial_j \varphi, \;\; \forall \varphi \in \mathcal{K}^\infty(\Omega).$$

(e) *Having* $q \in \mathcal{C}_\nabla(\Omega; E^d)$ *implies the properties of (c), but there exists open sets* Ω *for which the converse is false.* ▮

Case of a field with compact support. If the support of q is compact and included in Ω, the equivalences in (b) hold even when Ω is not simply connected.

Indeed, if q has local primitives on Ω, then its extension by 0_E has local primitives on the whole of $\mathbb{R}^d$, which is simply connected, so it has a primitive on the whole of $\mathbb{R}^d$ whose restriction is a primitive of q on Ω. Therefore, the properties in (c) are equivalent to those in (a). □

Proof of Theorem 9.14. (a) *First equivalence.* For every $f \in \mathcal{C}^1(\Omega; E)$ and every $\psi \in \mathcal{K}^\infty(\Omega; \mathbb{R}^d)$, the formula of integration by parts from Theorem 6.12 gives

$$\int_\Omega \nabla f \cdot \psi = \sum_{i=1}^d \int_\Omega \partial_i f \, \psi_i = -\sum_{i=1}^d \int_\Omega f \, \partial_i \psi_i = -\int_\Omega f \, \nabla \cdot \psi.$$

If $\nabla f = q$, we therefore have $\int_\Omega q \cdot \psi = -\int_\Omega f \, \nabla \cdot \psi = 0_E$ whenever $\nabla \cdot \psi = 0$. The converse is given by the orthogonality theorem (Theorem 9.2).

Second equivalence. If $q = \nabla f$, its line integral around closed paths is zero (Theorem 8.11 (b)). The converse is given by Theorem 9.1.

(c) *First equivalence.* If $\int_\Omega q_j \partial_i \varphi = \int_\Omega q_i \partial_j \varphi$ for all i, j and φ, and if B is a ball included in Ω, then Poincaré's theorem (Theorem 9.11) shows that, by restricting to the φ with support in B, q has a primitive f_B on this set.

Consider now the converse (which is straightforward when Ω is simply connected, see the comment *Optimality of Theorem 9.11*, p. 204, but here Ω is an arbitrary open set).

Let $\varphi \in \mathcal{K}^\infty(\Omega)$ with support K. Each point x of K is contained in an open ball B_x included in Ω, so, since K is compact, we can take a finite subcovering $(B_m)_{m \in M}$, where B_m denotes B_{x_m}. Let $(\alpha_m)_{m \in M}$ be a partition of unity subordinate to the covering by the B_m of their union ω, given by Theorem 7.18. The support of $q_j \partial_i \varphi$ is included in the support of φ and therefore certainly included in the open set ω, so its integral may be restricted to ω by Theorem 4.17 (a), that is,

$$\int_\Omega q_j \partial_i \varphi = \int_\omega q_j \partial_i \varphi.$$

Since $\sum_{m \in M} \alpha_m = 1$ on ω, it follows that

$$\int_\Omega q_j \partial_i \varphi = \int_\omega q_j \partial_i \Big(\sum_{m \in M} \alpha_m \varphi \Big) = \sum_{m \in M} \int_\omega q_j \partial_i (\alpha_m \varphi). \tag{9.13}$$

Suppose now that q has a primitive on each ball, and let f_m be a primitive on B_m. Since the support of $\alpha_m \varphi$ is included in B_m, together with Theorem 6.12 on integration by parts, it follows that

$$\int_\omega q_j \partial_i (\alpha_m \varphi) = \int_{B_m} \partial_j f_m \partial_i (\alpha_m \varphi) = -\int_{B_m} f_m \partial_j \partial_i (\alpha_m \varphi).$$

The derivatives commute by Schwarz's theorem (Theorem 2.12), so we can exchange i and j in this formula, and therefore in (9.13), which gives

$$\int_\Omega q_j \partial_i \varphi = \int_\Omega q_i \partial_j \varphi.$$

Second equivalence. If $q = \nabla f_B$ on each ball B included in Ω, the homotopy invariance theorem (Theorem 8.20) gives $\int_\Gamma q \cdot \mathrm{d}\ell = \int_{\Gamma_*} q \cdot \mathrm{d}\ell$ for every pair of homotopic closed paths Γ and Γ_* in Ω.

Conversely, suppose that this property is satisfied and let B be a ball included in Ω. Every closed path Γ of B is homotopic in this set to a closed path Γ_* consisting of a single point. Since the line integral is zero around Γ_* (Theorem 8.10 (a)), it is also zero around Γ, and therefore Theorem 9.1 gives f_B such that $q = \nabla f_B$ on B.

(b) If Ω is simply connected, $\nabla f = q$ is equivalent to $\nabla f_B = q$ on each B by Theorem 9.4 on gluing together local primitives, and therefore the properties of (a) are equivalent to those of (c).

(d) This is Theorem 9.9.

(e) If $\nabla f = q$ on Ω, then this certainly also holds on B. The converse is false, as shown by the examples that we will construct in Theorems 9.15, for $d = 2$, and 9.16, for arbitrary $d \geq 2$. □

9.8. Fields with local primitives but no global primitive

Let us show that there exist open sets on which the existence of local primitives does not guarantee the existence of a global primitive. Let us begin with an example in two dimensions with real values.

THEOREM 9.15.– *Let $\Omega = \{x \in \mathbb{R}^2 : |x| > 1\}$, and let $q \in \mathcal{C}^\infty(\Omega; \mathbb{R}^2)$ be the field defined for every $x \in \Omega$ by $q(x) = (-x_2, x_1)/|x|^2$.*

For every ball B included in Ω, there exists $f_B \in \mathcal{C}^\infty(B)$ such that $\nabla f_B = q$ on B. But there does not exist a function $f \in \mathcal{C}^1(\Omega)$ such that $\nabla f = q$ on the whole of Ω. ▮

Proof. In polar coordinates, $\nabla = \mathbf{e}_r \partial_r + (\mathbf{e}_\theta / r)\partial_\theta$ and $q(\theta, r) = \mathbf{e}_\theta / r$, so

$$\nabla\theta = q \ \text{ except at } \theta = 0.$$

Indeed, θ is discontinuous on the half-line $D = \{(r, \theta) : \theta = 0\}$, since it is equal to 0 on one side and 2π on the other.

The field q does not have a primitive, as this primitive would necessarily be continuous (Theorem 2.10) and its restriction to $\Omega \setminus D$ would be of the form $\theta + c$ (Theorem 2.7), which is a contradiction.

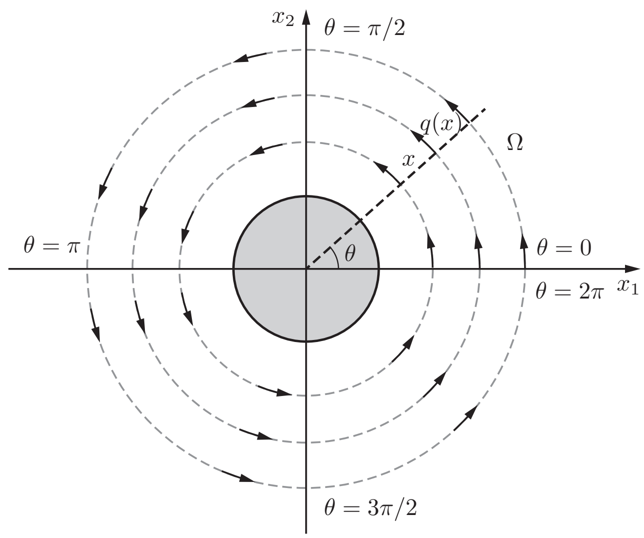

Figure 9.2. *Field q with local primitive θ but no global primitive. The set Ω is the exterior of the dashed disk*

Nevertheless, $\nabla\theta = q$ on every ball B because q is also a gradient on any ball that intersects with D, which can be seen by choosing another half-line as the origin of θ. $\square$

Let us show that such open sets also exist in arbitrary dimensions, $d \geq 2$.

THEOREM 9.16.– *Let $d \geq 2$, and let E be a Neumann space that is not just $\{0_E\}$.*

Then there exists an open subset Ω of $\mathbb{R}^d$ and a field $q \in \mathcal{C}^\infty(\Omega; E^d)$ such that: for every ball B included in Ω, there exists a function $f_B \in \mathcal{C}^\infty(B; E)$ such that $\nabla f_B = q$ on B, but there does not exist a function $f \in \mathcal{C}^1(\Omega; E)$ such that $\nabla f = q$ on the whole of Ω. ▌

Proof. Let Ω_2 be the open subset of $\mathbb{R}^2$ and $\mathbf{q}$ the field given in Theorem 9.15, and let $u \in E$, $u \neq 0_E$. In two dimensions, the field defined on Ω_2 by $q(x) = \mathbf{q}(x)\,u$ is as required. In dimensions higher than two, the field defined on $\Omega_2 \times \mathbb{R}^{d-2}$ by $q(x_1, \ldots, x_d) = (\mathbf{q}_1(x_1, x_2), \mathbf{q}_2(x_1, x_2), 0, \ldots, 0)\,u$ is as required. $\square$

Is simple connectedness necessary for gluing together local primitives? Recall that simple connectedness is sufficient for gluing together local primitives, that is, to guarantee that any field with local primitives has a global primitive by Theorem 9.4 and therefore to guarantee that a field satisfying Poincaré's condition, whether the strong or the weak variant, has a primitive.

Conversely, it is necessary if $d = 1$ or 2, but this is no longer the case when $d \geq 3$, although simple connectedness remains necessary for $d = 3$ if certain regularity conditions are imposed on the open set.

These results, which were communicated to me by Pierre DREYFUSS and Nicolas DEPAUW, appeal to difficult ideas from algebraic topology, presented in [DREYFUSS, 29]. An overview is given below.

Case where $d = 1$. Every open subset of $\mathbb{R}$ is simply connected and therefore has the local primitive gluing property.

Case where $d = 2$. Every open subset Ω of $\mathbb{R}^2$ that is not simply connected has at least one *hole*; in other words, there exists a point $z \notin \Omega$ enclosed by a closed path Γ in Ω. This set therefore does not have the local primitive gluing property, since the field q introduced in Theorem 9.15, after being translated in such a way that z is at the origin, is locally a gradient on Ω but globally not a gradient.

Case where $d = 3$. The exterior of the *Alexander horned sphere* (presented and studied in [HATCHER, 45, p. 171]) has the local primitive gluing property but is not simply connected.

However, for an open subset of $\mathbb{R}^3$ that is bounded and locally on one side of the graph of a continuous function, the local primitive gluing property implies simple connectedness.

Case where $d \geq 4$. The local primitive gluing property of an open subset of $\mathbb{R}^d$ does not imply simple connectedness, even for bounded open sets that are locally on one side of the graph of a continuous function. □

9.9. Uniqueness of primitives

Let us show that primitives are unique up to an additive constant on each connected component of the domain.

THEOREM 9.17.– *Let $q \in \mathcal{C}_\nabla(\Omega; E^d)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space. Then:*

(a) *The field q has infinitely many primitives.*

(b) *The set of all primitives may be deduced from a given primitive by adding arbitrary constants to each connected component Ω_m of Ω.*

(c) *Given a point $a_m \in \Omega_m$ and $c_m \in E$ for each connected component Ω_m of Ω, there exists one and only one primitive f such that: for every m,*

$$f(a_m) = c_m. \blacksquare$$

Reminder. We denote by $\mathcal{C}_\nabla(\Omega; E^d)$ the space of continuous fields with a primitive (Definition 9.13). □

Proof. (a) Every function $f + c$, where $c \in E$, is a primitive of q.

(b) If f and f' are two primitives, $\nabla(f' - f) = 0$, so, by Theorem 2.7, $f' - f$ is constant on each Ω_m. Conversely, $f + c$ is a primitive if c is constant on each Ω_m.

(c) If g is a primitive, then the function defined on each Ω_m by $f = g - g(a_m) + c_m$ is a primitive such that $f(a_m) = c_m$ for every m. This is the only one because, if f' is another primitive satisfying this property, then $f - f'$ is zero at a_m and is constant on Ω_m by (b) for every m, and is therefore zero on the whole of Ω. $\square$

Case of a connected open subset. If Ω is connected, then its only connected component is Ω itself, which simplifies the statement of parts (b) and (c) of Theorem 9.17. $\square$

9.10. Continuous primitive mapping

Let us construct a continuous linear primitive mapping.

THEOREM 9.18.– *Let $q \in \mathcal{C}_\nabla(\Omega; E^d)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space. For each connected component Ω_m of Ω, let $a_m \in \Omega_m$. Finally, let $f \in \mathcal{C}^1(\Omega; E)$ be the unique function such that*

$$\nabla f = q, \quad f(a_m) = 0_E, \ \forall m.$$

Then the mapping $q \mapsto f$ is linear and continuous and therefore sequentially continuous from $\mathcal{C}_\nabla(\Omega; E^d)$ into $\mathcal{C}^1(\Omega; E)$.

This mapping coincides with the mapping $q \mapsto q^*$ given by Theorem 9.1. ▮

Reminder. The space $\mathcal{C}_\nabla(\Omega; E^d)$ of continuous fields that have a primitive is endowed with the semi-norms of $\mathcal{C}(\Omega; E^d)$ by Definition 9.13. $\square$

Notation. We could denote this mapping by ∇^{-1}, that is, $\nabla^{-1} q \overset{\text{def}}{=} f$, or alternatively $\nabla^{-1} \overset{\text{def}}{=} q^*$. $\square$

Before giving the proof, let us state a consequence of sequential continuity.

THEOREM 9.19.– *Consider a sequence $(f_n)_{n\in\mathbb{N}}$ and a function f of $\mathcal{C}^1(\Omega; E)$, where Ω is an open subset of $\mathbb{R}^d$ and E is a Neumann space. For each connected component Ω_m of Ω, let $a_m \in \Omega_m$. Suppose that, as $n \to \infty$,*

$$\nabla f_n \to \nabla f \textit{ in } \mathcal{C}(\Omega; E^d)$$

and, for every m,

$$f_n(a_m) \to f(a_m) \textit{ in } E.$$

Then

$$f_n \to f \textit{ in } \mathcal{C}^1(\Omega; E).$$ ▮

Proof of Theorem 9.19. Let g_n and g be the functions of $\mathcal{C}^1(\Omega; E)$ defined on each Ω_m by $g_n = f_n - f_n(a_m)$ and $g = f - f(a_m)$. Theorem 9.18 shows that $g_n \to g$ in $\mathcal{C}^1(\Omega; E)$, which implies that $f_n \to f$ in $\mathcal{C}^1(\Omega; E)$. $\square$

Proof of Theorem 9.18. Let $y \in \Omega$, Ω_m the connected component of Ω containing y, Γ a piecewise $\mathcal{C}^1$ path joining a_m to y in Ω_m (such a path exists by Theorem 8.5), and $\epsilon > 0$ such that the ball $B(y, \epsilon) = \{x \in \mathbb{R}^d : |x - y| \leq \epsilon\}$ is included in Ω.

Let $x \in \mathring{B}(y, \epsilon)$ and Λ the rectilinear path joining y to x, i.e. the path defined on $[0, 1]$ by $\Lambda(t) = y + t(x - y)$. The path $\Gamma \overrightarrow{\cup} \Lambda$ joins a_m to x in Ω_m, so the expression of the line integral of a gradient (Theorem 8.11 (a)) and the expression of their concatenation (Definition 8.15) give

$$f(x) = f(a_m) + \int_{\Gamma \overrightarrow{\cup} \Lambda} \nabla f \cdot \mathrm{d}\ell = \int_\Gamma q \cdot \mathrm{d}\ell + \int_\Lambda q \cdot \mathrm{d}\ell.$$

Let $\{\| \ \|_{E;\nu} : \nu \in \mathcal{N}_E\}$ be the family of semi-norms of E. The bound on the semi-norms of the line integral from Theorem 8.12 (a) gives, for every $\nu \in \mathcal{N}_E$,

$$\|f(x)\|_{E;\nu} \leq (\gamma_\Gamma + \gamma_\Lambda) \sup_{z \in [\Gamma] \cup B(y,\epsilon)} \|q(z)\|_{E^d;\nu},$$

where $[\Gamma]$ is the image of Γ, $\gamma_\Gamma = \sup_{t_i \leq t \leq t_e} |\Gamma'(t)|$ and $\gamma_\Lambda = \sup_{0 \leq t \leq 1} |\Lambda'(t)|$, and so $\gamma_\Lambda = |x - y| \leq \epsilon$.

Consider now a compact set $K \subset \Omega$. The open sets $\mathring{B}(y, \epsilon)$ cover this set, so there exists (Definition A.17 (a)) a finite subcovering $\mathcal{R}$. Thus,

$$\sup_{x \in K} \|f(x)\|_{E;\nu} \leq c \sup_{z \in D} \|q(z)\|_{E^d;\nu},$$

where $c = \sup_{\mathcal{R}} \gamma_\Gamma + \epsilon$ is finite and $D = \bigcup_{\mathcal{R}} [\Gamma] \cup B(y, \epsilon)$. But D is compact, as a finite union of closed and bounded sets (which means that it is itself closed and bounded (Theorem A.10) in $\mathbb{R}^d$ and therefore compact by the Borel–Lebesgue theorem (Theorem A.23 (b))). And it is included in Ω. Therefore, by Definition 1.3 (a) of the semi-norms of $\mathcal{C}(\Omega; E)$, the above inequality can be written as

$$\|f\|_{\mathcal{C}(\Omega;E);K,\nu} \leq c \, \|q\|_{\mathcal{C}(\Omega;E^d);D,\nu}.$$

By Definition 2.14 (a) of the semi-norms of $\mathcal{C}^1(\Omega; E)$, since $K \subset D$ and $\partial_i f = q_i$,

$$\|f\|_{\mathcal{C}^1(\Omega;E);K,\nu} = \sup\{\|f\|_{\mathcal{C}(\Omega;E);K,\nu}, \sup_{1 \leq i \leq d} \|\partial_i f\|_{\mathcal{C}(\Omega;E);K,\nu}\} \leq$$
$$\leq \sup\{c, 1\} \|q\|_{\mathcal{C}(\Omega;E^d);D,\nu}.$$

By the characterization of continuous linear mappings from Theorem 1.25, this proves that the mapping $q \mapsto f$ is continuous. It is therefore sequentially continuous, like any continuous mapping (Theorem A.29). $\square$

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