A convection model for fire spread simulation

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Abstract

We present a convection model which can be coupled with fire propagation models in order to take into account the wind and the slope which are two of the most relevant factors affecting surface fire spread. An asymptotic analysis gives a three dimensional convective model governed by a two–dimensional equation.

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1 Introduction

In a complete fire spread model one should distinguish between the boundary layer where the fire takes place (reduced to a two–dimensional surface in a two–dimensional model) and the layer above where the air movement occurs. The main contribution of this paper is a model which provides a three–dimensional velocity wind field in the air layer under the influence of the fire, governed by a two–dimensional equation, so that it can be coupled with two–dimensional fire spread simulation models.

The validity of this model has the following limits: The nonlinear terms are neglected and we assume that the air temperature linearly decreases with the height. On the contrary the model takes into account buoyancy forces, slope effects and mass conservation.

The convection model is an adaptation to forest fires convective phenomena of the shallow water models proposed in Bresch [2].

2 Convection equations

Let $d \subset \Re^2$ be a two–dimensional normalized bounded domain, representing the projection of the three–dimensional geographical surface, $\mathbf{x}$ be any of its points and t be the time. We use small letters for the two–dimensional problem, and capital letters for the three–dimensional problem. All quantities are adimensionalised.

Let us consider the three–dimensional domain $D=\{(\mathbf{x},z) \colon \mathbf{x} \in d, h(\mathbf{x}) < z < \delta\}$ representing the air layer influenced by the fire. We assume that the height δ is small in front of the width and that the height of the surface at point $\mathbf{x}$, $h(\mathbf{x})$, is smaller than δ. We denote by an index $_{\mathbf{x}z}$ the three–dimensional operators, that is, $\nabla_{\mathbf{x}z} = (\partial_{x_1}, \partial_{x_2}, \partial_z)$ and $\Delta_{\mathbf{x}z} = \partial^2_{x_1 x_1} + \partial^2_{x_2 x_2} + \partial^2_{zz}$.

The air velocity $\mathbf{U} = (U_1, U_2, U_3)$ and the potential P satisfy the Navier–Stokes equations. On one hand, the momentum equation reads

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{\mathbf{x}z} \mathbf{U} - \frac{1}{Re} \Delta_{\mathbf{x}z} \mathbf{U} + \nabla_{\mathbf{x}z} P = \varphi Q \mathbf{e}_3 \tag{1}$$

where Q is the temperature, Re is the Reynolds number and $\mathbf{e}_3 = (0,0,1)$. The right–hand side represents the buoyancy forces due to the expansion under the effect of heat, which has the form $\varphi(Q - Q_0)$. Notice that the part of the force corresponding to the reference temperature Q_0 is $\varphi Q_0 \mathbf{e}_3 = \nabla_{\mathbf{x}z}(\varphi Q_0 z)$, that allows to include it into the potential term $\nabla_{\mathbf{x}z} P$. The density variations due to the temperature have been neglected in the other terms of the equation. On the other hand, the air compressibility is also neglected, so that

$$\nabla_{\mathbf{x}z} \cdot \mathbf{U} = 0. \tag{2}$$

In order to specify the boundary conditions, we decompose the boundary into $\partial D = S \cup A \cup L$, where the surface $S = \{(\mathbf{x}, z) : \mathbf{x} \in d,\ z = h(\mathbf{x})\}$, the air upper boundary $A = \{(\mathbf{x}, z) : \mathbf{x} \in d,\ z = \delta\}$ and the air lateral boundary $L = \{(\mathbf{x}, z) : \mathbf{x} \in \partial d,\ h(\mathbf{x}) < z < \delta\}$.

Boundary conditions are

$$\mathbf{U} \cdot \mathbf{N} = 0, \quad \frac{\partial \mathbf{U}}{\partial \mathbf{N}}\Big|_{tan} = \zeta \mathbf{U}, \quad \text{on } S, \tag{3}$$

$$\mathbf{U} \cdot \mathbf{N} = 0, \quad \frac{\partial \mathbf{U}}{\partial N}\Big|_{tan} = 0, \quad \text{on } A, \tag{4}$$

$$\mathbf{U}\big|_L = (\mathbf{v}_m, 0), \quad \text{on } L. \tag{5}$$

$\mathbf{N}$ is the inner unit normal vector field to ∂D, and the subscript $_{tan}$ denotes the tangential component, that is, $\mathbf{f}_{tan} = \mathbf{f} - (\mathbf{f} \cdot \mathbf{N})\,\mathbf{N}$ for any vector field $\mathbf{f}$. Since the boundary A in equation (4) is horizontal, this condition reads $U_3 = 0$ and $\partial_z U_1 = \partial_z U_2 = 0$. Equation (5) expresses that $\mathbf{U}(t, \mathbf{x}, z) = (\mathbf{v}_m(t, \mathbf{x}), 0)$ where $\mathbf{v}_m$ is the meteorological wind, that we assumed to be known, horizontal, non depending on z and with a null total flux through the lateral boundary, that is,

$$\partial_z \mathbf{v}_m = 0, \qquad \int_{\partial d} (\delta - h)\,\mathbf{v}_m \cdot \mathbf{n}\,ds = 0 \tag{6}$$

where $\mathbf{n} = (n_1, n_2)$ is the inner unit normal vector field to ∂d. We complete these equations with the initial condition

$$\mathbf{U}|_{t=0} = \mathbf{U}_0 \tag{7}$$

where $\mathbf{U}_0$ is the initial velocity, that we assume to be known. Equations (1) to (7) are well posed.

3 Horizontal flux

We distinguish the vertical velocity from the horizontal one denoting $\mathbf{V} = (U_1, U_2)$, $W = U_3$ and we define the horizontal flux at a point $\mathbf{x} \in d$ and time t by

$$\overline{\mathbf{V}}(t, \mathbf{x}) = \int_{h(\mathbf{x})}^{\delta} \mathbf{V}(t, \mathbf{x}, z)\,dz. \tag{8}$$

The incompressibility and the fact that the air does neither cross S nor A, that is, $\mathbf{U} \cdot \mathbf{N} = 0$, involve that the horizontal flux is also incompressible, then

$$\nabla_{\mathbf{x}} \cdot \overline{\mathbf{V}} = 0 \tag{9}$$

4 Convection asymptotic model

We now use the fact that thickness δ of the considered air layer, is small compared with its width. We also assume that the wind is not too strong and more precisely that $\delta^2 Re \ll 1$. Notice that this is similar to the Reynolds approximation, classic in lubrification, see Bayada [1]. Preserving only the dominant terms and re-scaling P, equations (1) and (2) give

$$-\partial^2_{zz}\mathbf{V} + \nabla_{\mathbf{x}} P = 0, \tag{10}$$
$$\partial_z P = \lambda Q, \tag{11}$$
$$\nabla_{\mathbf{x}} \cdot \mathbf{V} + \partial_z W = 0, \tag{12}$$

where $\lambda = \varphi Re$. Conditions (3), (4) and (5) particularly give

$$\partial_z \mathbf{V} = \zeta \mathbf{V}, \quad (\mathbf{V}, W) \cdot \mathbf{N} = 0, \qquad \text{on } S, \tag{13}$$
$$\partial_z \mathbf{V} = 0, \quad W = 0, \qquad \text{on } A, \tag{14}$$
$$\overline{\mathbf{V}} \cdot \mathbf{n} = (\delta - h)\mathbf{v}_m \cdot \mathbf{n}, \qquad \text{on } \partial d. \tag{15}$$

Equations (10) to (15) are well posed: for given Q and $\mathbf{v}_m$, there exists a unique solution $(\mathbf{V}, W, P)$ (up to an additive constant for P).

5 Solution of the convection asymptotic model

Let us explicitly compute $P(t, \mathbf{x}, z)$ and $\mathbf{V}(t, \mathbf{x}, z)$ in terms of z, $h(\mathbf{x})$, the temperature on the surface $q(t, \mathbf{x})$ and $\nabla_{\mathbf{x}} p(t, \mathbf{x})$, where p is a two–dimensional potential. We assume that the air temperature linearly decreases with the height and takes the value of the environmental temperature on the upper boundary of the air layer, that is

$$Q(t, \mathbf{x}, z) = q(t, \mathbf{x}) \frac{\delta - z}{\delta - h(\mathbf{x})} \tag{16}$$

For a fixed $\mathbf{x}$, equation (11) with Q given by (16), provides for some $p(t, \mathbf{x})$:

$$P(t, \mathbf{x}, z) = p(t, \mathbf{x}) + \frac{\lambda q(t, \mathbf{x})}{\delta - h(\mathbf{x})}(\delta z - \frac{1}{2} z^2) \tag{17}$$

Equation (10), together with conditions $\partial_z \mathbf{V}(t, \mathbf{x}, \delta) = 0$ and $\partial_z \mathbf{V}(t, \mathbf{x}, h(\mathbf{x})) = \zeta \mathbf{V}(t, \mathbf{x}, h(\mathbf{x}))$ included in (14) and (13), provide

$$\begin{aligned} \mathbf{V}(t, \mathbf{x}, z) = &\Big(\frac{1}{2} z^2 - \delta z - \frac{1}{2} h^2(\mathbf{x}) + (\delta + \xi) h(\mathbf{x}) - \xi \delta\Big) \nabla_{\mathbf{x}} p(t, \mathbf{x}) \\ &+ \Big(- \frac{1}{24} z^4 + \frac{1}{6} \delta z^3 - \frac{1}{3} \delta^3 z + \frac{1}{24} h^4(\mathbf{x}) - \frac{1}{6} h^3(\mathbf{x})(\delta + \xi) \end{aligned}$$

$$+\frac{1}{2}\xi\delta h^2(\mathbf{x}) + \frac{1}{3}\delta^3 h(\mathbf{x}) - \frac{1}{3}\xi\delta^3\Big)\nabla_{\mathbf{x}}\hat{q}(t,\mathbf{x}) \tag{18}$$

where $\xi = 1/\zeta$, $\hat{q}(t,\mathbf{x}) = \lambda q(t,\mathbf{x})/(\delta - h(\mathbf{x}))$. Notice that $\hat{q}(t,\mathbf{x}) = (\lambda/\delta)\,Q(t,\mathbf{x},0)$, where $Q(t,\mathbf{x},0)$ is the temperature at reference height $z = 0$.

6 Stream function 2D equations

We now assume that ∂d is connected. Since it is bounded, then d is simply connected and it is "within" its boundary ∂d. The hypothesis (6), that is $\int_{\partial d}(\delta - h)\,\mathbf{v}_m \cdot \mathbf{n}\,ds = 0$, gives the existence of a function $\mathbf{v}^*$ such that

$$\nabla_{\mathbf{x}} \cdot \mathbf{v}^* = 0, \quad \mathbf{v}^* \cdot \mathbf{n} = (\delta - h)\,\mathbf{v}_m \cdot \mathbf{n} \text{ on } \partial d. \tag{19}$$

From (9) it follows that $\nabla_{\mathbf{x}} \cdot (\overline{\mathbf{V}} - \mathbf{v}^*) = 0$ therefore, again thanks to (19), there exists a function k such that

$$\nabla_{\mathbf{x}}^{\perp} k = \overline{\mathbf{V}} - \mathbf{v}^*, \quad k|_{\partial d} = 0, \tag{20}$$

where $\nabla_{\mathbf{x}}^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ is the usual 2D **rot** operator. Integrating (18) with respect to z, from $h(\mathbf{x})$ to δ, we obtain (definition(8))

$$\overline{\mathbf{V}} = -a\nabla_{\mathbf{x}} p - b\nabla_{\mathbf{x}}\hat{q} \tag{21}$$

where $a(\mathbf{x}) = \frac{1}{3}(\delta - h(\mathbf{x}))^2(3\xi + \delta - h(\mathbf{x}))$ and $b(\mathbf{x}) = \frac{1}{30}(\delta - h(\mathbf{x}))^2(2\delta^2(2\delta + 5\xi) - 2\delta(\delta - 5\xi)h(\mathbf{x}) - (3\delta + 5\xi)h^2(\mathbf{x}) + h^3(\mathbf{x}))$.

This implies for $f = \nabla_{\mathbf{x}}^{\perp} \cdot \frac{1}{a}(b\nabla_{\mathbf{x}}\hat{q} + \mathbf{v}^*)$,

$$-\nabla_{\mathbf{x}} \cdot \Big(\frac{1}{a}\nabla_{\mathbf{x}} k\Big) = f, \quad k|_{\partial d} = 0, \tag{22}$$

7 Velocity on the surface

The velocity on the surface is $\mathbf{v}(t,\mathbf{x})$=$\mathbf{V}(t,\mathbf{x},h(\mathbf{x}))$, then (18), (20), (21) give

$$\mathbf{v}(t,\mathbf{x}) = \frac{\xi}{c(\mathbf{x})}(\nabla_{\mathbf{x}}^{\perp} k(t,\mathbf{x}) + e(\mathbf{x})\nabla_{\mathbf{x}}\hat{q}(t,\mathbf{x}) + \mathbf{v}^*(t,\mathbf{x})), \tag{23}$$

where $c(\mathbf{x}) = \frac{1}{3}(\delta - h(\mathbf{x}))\,(\delta + 3\xi - h(\mathbf{x}))$ and $e(\mathbf{x}) = \frac{1}{45}(\delta - h(\mathbf{x}))^5$.

8 Application, validity of the asymptotic model and conclusions

Fire propagation is frequently modelled by a 2D convection-diffusion-reaction equation on the temperature q, see Asensio [3],

$$\partial_t q + \mathbf{v}.\nabla_{\mathbf{x}} q - \nabla_{\mathbf{x}}.(\kappa(q)\nabla_{\mathbf{x}} q) = s(q) \tag{24}$$

where $\kappa(q)$ is a diffusion term which takes into account local radiation and $s(q)$ is the heat source term due to combustion. The temperature is computed solving (24) step by step. In each time step the velocity $\mathbf{v}$ in the convection term is computed explicitly by expression (23), solving first the 2D elliptic problem (22) to obtain the potential k.

The previous asymptotic model can be mathematically justified when Reynolds number is small enough, considering turbulent viscosity. This is equivalent to replace the nonlinear transport terms by linear diffusion terms. Briefly, the whirlwinds of size lower than the height of the air layer perturbed by the fire are removed. We obtain a velocity field explicitly computable in terms of z. It is lost a part of the effects due to the turbulence that can be interpreted as instabilities (or multiple bifurcations) due to the nonlinearity.

The obtained approximation is reasonable because it takes into account, on one hand, buoyancy force due to the air expansion under the effect of heat, and on the other hand, the local movement transmission generated in the fluid set under the effect of the incompressibility. These two phenomena roughly determine the air movement.

The main contribution of this model is to develop a method which provides a three-dimensional velocity wind field in the air layer under the influence of the fire, which is governed by a bi-dimensional equation verified by a stream function. More precisely, we compute explicitly the three-dimensional air velocity as a function of the vertical coordinate z, the stream function k, the surface temperature q and the surface height h.

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