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# High Definition local adjustment model of 3D Wind Fields performing only 2D computations

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## SUMMARY

We present a set of equations modelling wind velocity in a 3D domain in terms of the ground height function, the ground temperature and the wind on the boundary. The wind field is adjusted to several punctual wind velocity measurements at different points in the 3D domain by an optimal control

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problem in which the (unknown) wind on the boundary is the control. Using the meteorological wind punctual measures as datum, the model provides locally a detailed 3D wind which takes into account topography and thermal gradients on the surface by solving only 2D linear equations. We consider briefly the numerical approximation and two examples, including one with real data to control the accuracy of the model. Copyright © 200 John Wiley & Sons, Ltd.

KEY WORDS: wind field adjustment; optimal control

## 1. INTRODUCTION

Wind models are important tools that allow the study of several problems related to the atmosphere, such as the effect of wind on structures, wind power, pollutant transport, fire spreading [1], and so forth. Our starting point is a mass consistent vertical diffusion wind field model. If the significant phenomena that we want to simulate occur in a zone where the horizontal dimensions are much larger than the vertical one, then an asymptotic approximation of the primitive Navier–Stokes equations is derived as in [2].

The most salient feature of this asymptotic approximate is that it provides a three-dimensional velocity wind field governed by a two-dimensional linear equation. This field satisfies the incompressibility condition in the air layer and it takes into account the terrain elevation information and the temperature surface distribution in order to model thermal effects such as sea breezes. The validity of this model has the following limits: the nonlinear terms are neglected, and it is assumed that the air temperature decreases linearly with the height. In addition, the air compressibility is neglected. On the other hand, the model takes into account buoyancy forces, slope effects, and mass conservation.

The wind being only known by measurements at some points of the 3D domain, the solution

of the asymptotic model is adjusted to these data by an optimal control problem which is solved using the adjoint equation-based method.

The outlines of the paper are as follows. In Section 2 we describe the general notation and the wind field mathematical model leading to an optimal control problem, that is derived and characterized in Section 3. In Section 4 we establish the numerical approximation and describe briefly the solution methods for the linear system. In Section 5 we report on numerical experiments. Some conclusions are given in Section 6.

## 2. WIND FIELD MODEL

### 2.1. Wind equations

Let  $\omega \subset \mathbb{R}^2$  be a two-dimensional normalized bounded and connected domain representing the projection of the three-dimensional ground surface,  $x = (x_1, x_2)$  be any of its points and  $\tau$  be the time. We use small letters for the two-dimensional problem, and capital letters for the three-dimensional problem. All quantities are adimensionalised.

Let us consider the three-dimensional domain  $\Omega = \{(x, z) : x \in \omega, h(x) < z < \delta\}$  representing the air layer studied. We assume that the height  $\delta$  is small in front of the width and that the height  $h(x)$  of the surface at point  $x$  is smaller than  $\delta$ . In this section, we denote by an index  $xz$  the three-dimensional operators, and by an index  $x$  the bi-dimensional operators.

The air velocity  $U = (U_1, U_2, U_3)$  and the potential  $P$  satisfy the Navier–Stokes equations. On one hand, the momentum equation reads

$$\frac{\partial U}{\partial \tau} + U \cdot \nabla_{xz} U - \frac{1}{Re} \Delta_{xz} U + \nabla_{xz} P = \frac{\lambda}{Re} T e_3, \quad (1)$$

where  $T$  is the adimensional temperature,  $Re$  is the Reynolds number,  $\lambda$  is related to the

Grashoff number (indeed,  $\lambda T$  is the Grashoff number), and  $e_3 = (0, 0, 1)$ .

On the other hand, the air compressibility is neglected, so that

$$\nabla_{xz} \cdot U = 0. \quad (2)$$

In order to specify the boundary conditions, we decompose the boundary into  $\partial\Omega = S \cup A \cup L$ , where  $S = \{(x, z) : x \in \omega, z = h(x)\}$  is the ground surface,  $A = \{(x, z) : x \in \omega, z = \delta\}$  is the air upper boundary and  $L = \{(x, z) : x \in \partial\omega, h(x) < z < \delta\}$  is the air lateral boundary.

Boundary conditions are

$$U \cdot N = 0, \quad \frac{\partial U}{\partial N} \Big|_{tan} = \zeta U, \quad \text{on } S, \quad (3)$$

$$U_3 = 0, \quad \partial_z U_1 = \partial_z U_2 = 0, \quad \text{on } A, \quad (4)$$

$$U|_L = (v_m, 0), \quad \text{on } L, \quad (5)$$

where  $\zeta$  is the friction coefficient,  $N$  is the 3D inner unit normal vector field to  $\partial\Omega$ , and the subscript  $tan$  denotes the tangential component. Here  $v_m$  is the horizontal component of the meteorological wind (its vertical component is neglected), that is assumed to be known, non depending on  $z$  and with a null total flux through the lateral boundary, that is,

$$\partial_z v_m = 0, \quad \int_{\partial\omega} (\delta - h) v_m \cdot \nu \, ds = 0, \quad (6)$$

where  $\nu = (\nu_1, \nu_2)$  is the 2D inner unit normal vector field to  $\partial\omega$ . We complete these equations with the initial condition

$$U|_{\tau=0} = U_0, \quad (7)$$

where  $U_0$  is the initial velocity, that we assume to be known. Equations (1) to (7) are well posed, *i.e.* there exists a solution  $(U, P)$ , which is unique up to an additive constant on pressure for a small enough Reynolds number.

We distinguish the vertical velocity from the horizontal one denoting  $W = U_3$ ,  $V = (U_1, U_2)$ , and we define the horizontal flux at a point  $x \in \omega$  and time  $\tau$  by

$$\bar{V}(\tau, x) = \int_{h(x)}^{\delta} V(\tau, x, z) dz. \quad (8)$$

The incompressibility and the fact that the air does neither cross  $S$  nor  $A$ , that is,  $U \cdot N = 0$ , involve that the horizontal flux is also incompressible, that is

$$\nabla_x \cdot \bar{V} = 0. \quad (9)$$

## 2.2. Asymptotic wind equations

Using the fact that the thickness  $\delta$  of the considered air layer is small compared with its width, and assuming that the wind is not too strong and more precisely that  $\delta^2 Re \ll 1$ , then preserving only the dominant terms and re-scaling  $P$ , Equations (1) and (2) write

$$-\partial_{zz}^2 V + \nabla_x P = 0, \quad (10)$$

$$\partial_z P = \lambda T, \quad (11)$$

$$\nabla_x \cdot V + \partial_z W = 0. \quad (12)$$

Boundary conditions (3), (4) and (5) give

$$(V, W) \cdot N = 0, \quad \partial_z V = \zeta V, \quad \text{on } S, \quad (13)$$

$$W = 0, \quad \partial_z V = 0, \quad \text{on } A, \quad (14)$$

$$\bar{V} \cdot \nu = (\delta - h)v_m \cdot \nu, \quad \text{on } \partial\omega. \quad (15)$$

Equations (10) to (15) are well posed: given  $v_m$  satisfying (6) and  $T$ , there exists a solution  $(V, W, P)$  which is unique up to an additive constant on  $P$ . For more details about this convection asymptotic model and its further solution (16), (17) and (18), see [2].

### 2.3. 3D Horizontal velocity in terms of 2D potential

We assume that the air temperature linearly decreases with the height and vanishes on the upper boundary (for a non zero constant given air temperature on the upper boundary, similar formulas hold and numerical results are very close), that is

$$T(\tau, x, z) = t_S(\tau, x) \frac{\delta - z}{\delta - h(x)},$$

and that the temperature  $t_S = t_S(\tau, x)$  on ground surface is given. Then, by Equation (11), there exists a 2D potential  $p = p(\tau, x)$  such that

$$P(\tau, x, z) = p(\tau, x) + t(\tau, x) \left( \delta z - \frac{1}{2} z^2 \right),$$

where  $t$  is a re-scaled temperature given by

$$t(\tau, x) = \frac{\lambda t_S(\tau, x)}{\delta - h(x)}.$$

The potential  $p$  is called 2D because it only depends on the first two spaces variables  $x = (x_1, x_2)$ ; it also depends on time  $\tau$ , but this last acts now as a parameter.

Then, Equation (10) and boundary conditions  $\partial_z V(\tau, x, \delta) = 0$  and  $\partial_z V(\tau, x, h(x)) = \zeta V(\tau, x, h(x))$  included in (14) and (13) provide

$$V(x, z) = m(x, z) \nabla_x p(\tau, x) + n(x, z) \nabla_x t(\tau, x), \quad (16)$$

where

$$\begin{aligned} m(x, z) &= \left( \frac{1}{2} z^2 - \delta z - \frac{1}{2} h^2(x) + (\delta + \xi) h(x) - \xi \delta \right), \\ n(x, z) &= \left( -\frac{1}{24} z^4 + \frac{1}{6} \delta z^3 - \frac{1}{3} \delta^3 z + \frac{1}{24} h^4(x) - \frac{1}{6} (\delta + \xi) h^3(x) \right. \\ &\quad \left. + \frac{1}{2} \xi \delta h^2(x) + \frac{1}{3} \delta^3 h(x) - \frac{1}{3} \xi \delta^3 \right), \end{aligned}$$

being  $\xi = 1/\zeta$  the inverse of the friction coefficient  $\zeta$ .

#### 2.4. Equations governing the 2D potential

Using (16) in Equations (9) (which follows from (12), (13) and (14)) and (15), we find that the potential  $p = p(\tau, x)$  satisfies the following 2D boundary problem

$$-\nabla_x \cdot (a \nabla_x p) = \nabla_x \cdot (b \nabla_x t) \quad \text{in } \omega, \quad (17)$$

$$a \frac{\partial p}{\partial \nu} = -b \frac{\partial t}{\partial \nu} + v \quad \text{on } \partial\omega, \quad (18)$$

where

$$a = a(x) = \frac{1}{3}(\delta - h(x))^2(3\xi + \delta - h(x)),$$

$$b = b(x) = \frac{1}{30}(\delta - h(x))^2 \left( 2\delta^2(2\delta + 5\xi) - 2\delta(\delta - 5\xi)h(x) - (3\delta + 5\xi)h^2(x) + h^3(x) \right),$$

and

$$v = v(\tau, x) = (\delta - h(x))v_m(\tau, x) \cdot \nu(x)$$

is the horizontal normal flux on  $\partial\omega$ . By Hypothesis (6), it satisfies

$$\int_{\partial\omega} v = 0. \quad (19)$$

Equations (17)–(18) are well posed: given  $t$  and  $v$ , there exists a solution  $p$ , which is unique up to an additive constant. This constant will be fixed by the extra condition

$$\int_{\omega} p = 0.$$

#### 2.5. Variational solution of the 2D potential equations

Let

$$L_0^2(\partial\omega) = \left\{ v \in L^2(\partial\omega) : \int_{\partial\omega} v = 0 \right\},$$

$$\mathcal{V} = \left\{ \varphi \in H^1(\omega) : \int_{\omega} \varphi = 0 \right\}.$$

Given  $t \in H^1(\omega)$  and  $v \in L_0^2(\partial\omega)$ , by Lax–Milgram theorem, there exists a unique  $p \in \mathcal{V}$  such that

$$\int_{\omega} a \nabla p \cdot \nabla \varphi = \int_{\partial\omega} v \varphi - \int_{\omega} b \nabla t \cdot \nabla \varphi, \quad \forall \varphi \in \mathcal{V}. \quad (20)$$

Let us check that it satisfies (17) and (18). Given  $\phi \in \mathcal{D}(\omega)$ ,  $\varphi = \phi + c$  belongs to  $\mathcal{V}$  for some  $c \in \mathbb{R}$ ; then (20) writes  $\int_{\omega} a \nabla p \cdot \nabla \phi = - \int_{\omega} b \nabla t \cdot \nabla \phi$  since  $\int_{\partial\omega} v(\phi + c) = c \int_{\partial\omega} v = 0$ . This provides Equation (17) in distribution sense. Now (17) and (20) give  $\int_{\omega} (\nabla \cdot (a \nabla p + b \nabla t)) \varphi = \int_{\partial\omega} v \varphi$ . This provides Boundary Condition (18) in a weak sense since  $\varphi$  may coincide with any given element of  $H^1(\omega)$  in a neighborhood of  $\partial\omega$ .

In fact, data depend on time  $\tau$  which acts as a parameter: given  $t(\tau) \in H^1(\omega)$  and  $v(\tau) \in L_0^2(\partial\omega)$ , there exists a unique solution  $p(\tau) \in \mathcal{V}$ .

### 3. THE OPTIMAL CONTROL PROBLEM

In order to simplify notation in the sequel, the time  $\tau$  is omitted since it acts as a parameter, and the index  $x$  is omitted in differential operators since they only act on this variable.

#### 3.1. Identification of wind on the boundary

We are going now to identify  $v$  to the solution  $u$  of an optimal control problem: given  $N$  experimental measurements  $V_i$  of the wind velocity at given points  $(x_i, z_i)$ ,  $i = 1, \dots, N$ , we search the value of  $v \in L_0^2(\partial\omega)$  such that the  $V(x_i, z_i)$  given by (16) are as close as possible to the experimental values  $V_i$ . This is an optimal control problem where:

- $v \in L_0^2(\partial\omega)$  is the control.
- The state equations is (20).

- The cost function is chosen to be

$$J(v) = \frac{1}{2} \sum_{i=1}^N \|V(x_i, z_i) - V_i\|^2 + \frac{\alpha}{2} \int_{\partial\omega} v^2.$$

Due to (16),

$$J(v) = \frac{1}{2} \sum_{i=1}^N \|m(x_i, z_i)\nabla p(x_i) + n(x_i, z_i)\nabla t(x_i) - V_i\|^2 + \frac{\alpha}{2} \int_{\partial\omega} v^2. \quad (21)$$

In practice, instead of (21), we use the following regularized functional

$$J(v) = \frac{1}{2} \sum_{i=1}^N \int_{\omega} \rho_{\epsilon, i}(x) \|m(x, z_i)\nabla p(x) + n(x, z_i)\nabla t(x) - V_i\|^2 dx + \frac{\alpha}{2} \int_{\partial\omega} v^2 \quad (22)$$

where  $\rho_{\epsilon, i}$  is a suitable mollifier cancelling outside a small ball  $B(x_i, \epsilon)$  and such that  $\int \rho_{\epsilon, i}(x) dx = 1$ . For example,

$$\rho_{\epsilon, i}(x) = \frac{1}{\epsilon^2} \rho\left(\frac{x - x_i}{\epsilon}\right),$$

where  $\rho(x) = c \exp(-1/(1 - \|x\|^2))$  for  $\|x\| < 1$ , and  $\rho(x) = 0$  for  $\|x\| \geq 1$ .

The optimal control problem to be solved is to find  $u \in L_0^2(\partial\omega)$  such that

$$J(u) = \inf_{v \in L_0^2(\partial\omega)} J(v). \quad (23)$$

*Remark.* The regularization term  $\frac{\alpha}{2} \int_{\partial\omega} v^2 d\sigma$  is necessary for mathematical reasons. Indeed, without this term the optimal control problem has no solution (if we search for the control  $u$  in the whole space  $L_0^2$ ). We have chosen  $\alpha$  a small number. If a good estimation of the wind flux  $v \approx u^*$  is known on the boundary, we can modify the cost functional in (21) as follows:

$$J(v) = \frac{1}{2} \sum_{i=1}^N \int_{\omega} \rho_{\epsilon, i}(x) \|m(x, z_i)\nabla p(x) + n(x, z_i)\nabla t(x) - V_i\|^2 dx + \frac{\alpha}{2} \int_{\partial\omega} (v - u^*)^2 d\sigma$$

and choose a larger value for  $\alpha$ . Note that we cannot impose at the same time the value of the wind flux on the boundary and the value of the solution at several given points, as once the wind flux  $v$  is given on the boundary the wind field is uniquely determined by (17)-(18).

Therefore the optimal control problem with the former functional is a compromise (and so is the value of the parameter  $\alpha$ ) between these two sets of data. Usually in practical applications we do not have a good estimation of the flux on the boundary, and this is the reason why we choose the value of  $\alpha$  to be small, typically  $\alpha = 0.001$ .

### 3.2. Solution of the optimal control problem

Let  $Bv \in \mathcal{V}$  be the solution of

$$\int_{\omega} a \nabla Bv \cdot \nabla \varphi = \int_{\partial\omega} v \varphi, \quad \forall \varphi \in \mathcal{V}, \quad (24)$$

and  $r \in \mathcal{V}$  be the solution of

$$\int_{\omega} a \nabla r \cdot \nabla \varphi = - \int_{\omega} b \nabla t \cdot \nabla \varphi, \quad \forall \varphi \in \mathcal{V}. \quad (25)$$

The operator  $B$  is linear continuous from  $L_0^2(\partial\omega)$  in  $\mathcal{V}$ , and  $r$  is independent of  $v$ .

The solution of (20) is  $p = Bv + r$ , therefore

$$J(v) = \frac{1}{2} \pi(v, v) - \lambda(v) + \mu \quad (26)$$

where  $\pi$  is the bilinear continuous function on  $L_0^2(\partial\omega) \times L_0^2(\partial\omega)$  given by

$$\pi(u, v) = \sum_{i=1}^N \int_{\omega} \rho_{\epsilon, i} (m \nabla Bu) \cdot (m \nabla Bv) + \alpha \int_{\partial\omega} uv, \quad (27)$$

$\lambda$  is the continuous linear function in  $L_0^2(\partial\omega)$  given by

$$\lambda(v) = \sum_{i=1}^N \int_{\omega} \rho_{\epsilon, i} (m \nabla r + n \nabla t - V_i) \cdot m \nabla Bv, \quad (28)$$

and  $\mu$  is the constant

$$\mu = \frac{1}{2} \sum_{i=1}^N \int_{\omega} \rho_{\epsilon, i} V_i \cdot V_i. \quad (29)$$

Obviously,

$$\pi(v, v) \geq \alpha \|v\|_{L_0^2(\partial\omega)}^2.$$

Then, by Theorem 1.1 Chapter 1 in [3], there exist a unique solution  $u$  of Problem (23). It is characterized by  $J'(u) = 0$ , where  $J'(u)$  is the derivative of  $J$  at point  $u$ , which is a real linear continuous function on  $L_0^2(\partial\omega)$ . That is

$$J'(u) \cdot v = 0, \quad \forall v \in L_0^2(\partial\omega). \quad (30)$$

Using the chain rule,

$$J'(u) \cdot v = \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} (m\nabla p + n\nabla t - V_i) \cdot m\nabla Bv + \alpha \int_{\partial\omega} uv \quad (31)$$

(this formula holds for all  $u \in \mathcal{V}$ ).

### 3.3. Definition of the adjoint state and elimination of the control

Denote now  $p$  the potential associated to  $u$  by (20), that is the unique solution  $p \in \mathcal{V}$  of

$$\int_{\omega} a\nabla p \cdot \nabla \varphi = \int_{\partial\omega} u\varphi - \int_{\omega} b\nabla t \cdot \nabla \varphi, \quad \forall \varphi \in \mathcal{V}. \quad (32)$$

Let

$$\mathcal{W} = \left\{ \psi \in H^1(\omega) : \int_{\partial\omega} \psi = 0 \right\}.$$

We introduce the adjoint state as the solution  $q \in \mathcal{W}$  of:

$$\int_{\omega} a\nabla q \cdot \nabla \psi = \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} (m\nabla p + n\nabla t - V_i) \cdot m\nabla \psi, \quad \forall \psi \in \mathcal{W}. \quad (33)$$

Since  $J'(u) = 0$  the expression (31) of  $J'$  provides

$$\sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} (m\nabla p + n\nabla t - V_i) \cdot m\nabla Bv = -\alpha \int_{\partial\omega} uv. \quad (34)$$

Let us take  $\psi = Bv + c$  in (33) where  $c = -\int_{\partial\omega} Bv/|\omega|$  is such that  $\psi \in \mathcal{W}$ . With (34) it gives

$$\int_{\omega} a\nabla q \cdot \nabla Bv = -\alpha \int_{\partial\omega} uv. \quad (35)$$

On the other hand, let us take  $\varphi = q + c'$  in Definition (24) of  $Bv$ , where  $c' = -\int_{\omega} q/|\omega|$  is such that  $\varphi \in \mathcal{V}$ . Since  $\int_{\partial\omega} v = 0$ , we get

$$\int_{\omega} a\nabla Bv \cdot \nabla q = \int_{\partial\omega} vq + c' \int_{\partial\omega} v = \int_{\partial\omega} vq. \quad (36)$$

Comparing (35) and (36), it results, for all  $v \in L_0^2(\partial\omega)$ ,

$$\int_{\partial\omega} qv = -\alpha \int_{\partial\omega} uv. \quad (37)$$

Since  $u \in L_0^2(\partial\omega)$  (by definition (23)) and  $q|_{\partial\omega} \in L_0^2(\partial\omega)$  (because  $\int_{\partial\omega} q = 0$  from  $q \in \mathcal{W}$ ),

(37) implies

$$u = -\frac{1}{\alpha}q. \quad (38)$$

Using this, Definition (32) of  $p$ , gives

$$\int_{\omega} a\nabla p \cdot \nabla \varphi + \frac{1}{\alpha} \int_{\partial\omega} q\varphi = - \int_{\omega} b\nabla t \cdot \nabla \varphi, \quad \forall \varphi \in \mathcal{V}. \quad (39)$$

#### 3.4. An equivalent problem

In order to allow easier numerical computation, let us see that it is equivalent to solve coupled equations (39) and (33) in  $H^1(\omega)$  instead of  $\mathcal{V}$  and  $\mathcal{W}$ .

On one hand,  $(p, q) \in H^1(\omega) \times H^1(\omega)$  and satisfies

$$\int_{\omega} a\nabla p \cdot \nabla \varphi + \frac{1}{\alpha} \int_{\partial\omega} q\varphi = - \int_{\omega} b\nabla t \cdot \nabla \varphi, \quad \forall \varphi \in H^1(\omega), \quad (40)$$

$$\int_{\omega} a\nabla q \cdot \nabla \psi = \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} (m\nabla p + n\nabla t - V_i) \cdot m\nabla \psi, \quad \forall \psi \in H^1(\omega). \quad (41)$$

Indeed (40) is satisfied by constant  $\varphi$  (since  $\int_{\partial\omega} q = 0$ ) and therefore by all  $\varphi' + c$  with  $\varphi' \in \mathcal{V}$  and  $c \in \mathbb{R}$ , that is for all  $\varphi \in H^1(\omega)$ . And (41) is satisfied by constant  $\psi$  and therefore by all  $\psi' + c$  with  $\psi' \in \mathcal{W}$ , that is for all  $\psi \in H^1(\omega)$ .

On the other hand, let us check that the solution of (40)–(41) is unique up to a constant on  $p$ . Let here  $(p, q)$  denote the difference of two possible solutions. It satisfies (40)–(41) with

$t = 0$  and  $V_i = 0$ . Setting  $\varphi = q$  in such (40) and  $\psi = -p$  in such (41) and adding, we get

$$\frac{1}{\alpha} \int_{\partial\omega} q^2 + \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} \|m\nabla p\|^2 = 0$$

and then  $q|_{\partial\omega} = 0$ . Setting now  $\varphi = p$  in (40), we get  $\int_{\omega} a \|\nabla p\|^2 = 0$  and then  $\nabla p = 0$  and  $p$  is constant. Setting finally  $\psi = q$  in (41), we get  $\int_{\omega} a \|\nabla q\|^2 = 0$  and then  $q|_{\partial\omega} = 0$  gives  $q = 0$ .

This proves that  $(\nabla p, q)$  is unique, that is the uniqueness of  $(p, q)$  up to a constant on  $p$ .

### 3.5. Expression of the adjusted wind

There exists a unique  $(\nabla p, q)$  solution of coupled equations (40)–(41). Indeed there exists  $u \in L_0^2(\partial\omega)$  satisfying (30), and then  $p \in \mathcal{V}$  solution of (32) and  $q \in \mathcal{W}$  solution of (33), and then  $(\nabla p, q)$  satisfies (40)–(41) and is unique as seen in section 3.4. The adjusted wind velocity  $V(x, z)$  is given in terms of this  $\nabla p(x)$  at every point  $(x, z)$  by (16).

## 4. APPROXIMATED SOLUTION

### 4.1. Approximated equations

In order to make  $p$  unique and to improve numerical solution, we actually compute the solution of the following approximated equations: find  $(p, q) \in H^1(\omega) \times H^1(\omega)$  solution of

$$\int_{\omega} a \nabla p \cdot \nabla \varphi + \eta \int_{\partial\omega} p \varphi + \frac{1}{\alpha} \int_{\partial\omega} q \varphi = - \int_{\omega} b \nabla t \cdot \nabla \varphi, \quad \forall \varphi \in H^1(\omega), \quad (42)$$

$$\int_{\omega} a \nabla q \cdot \nabla \psi + \eta \int_{\partial\omega} q \psi = \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} (m \nabla p + n \nabla t - V_i) \cdot m \nabla \psi, \quad \forall \psi \in H^1(\omega), \quad (43)$$

for a small parameter  $\eta$  (in the examples below  $\eta = 0.001$ ), and then we compute the (approximated adjusted) wind velocity  $V(x, z)$  at every point  $(x, z)$  in terms of  $\nabla p(x)$  using expression (16).

*Remark.* The reader could expect regularization terms involving integrals on  $\omega$  instead of  $\partial\omega$ . In (43), the regularization term  $\eta \int_{\partial\omega} q\psi$  is used in order to get  $\int_{\partial\omega} q = 0$  (by choosing  $\psi = 1$ ). Indeed this important condition (it gives  $u \in L_0^2(\partial\omega)$  by (38)) which followed from the exact equation (40) (by choosing  $\varphi = 1$ ) is no longer contained in the approximated equation (42). In this last, we then use the regularization term  $\eta \int_{\partial\omega} p\varphi$  in order to get Equality (44) below.

#### 4.2. Convergence

Let us check that the solution of (42)–(43) goes to the solution of exact equations (40)–(41) as  $\eta \rightarrow 0$ . At first, let us bound  $q$ . Setting  $\varphi = q$  in (42),  $\psi = -p$  in (43) and adding, we get

$$\frac{1}{\alpha} \int_{\partial\omega} q^2 + \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} \|m\nabla p\|^2 = - \int_{\omega} b\nabla t \cdot \nabla q - \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} (n\nabla t - V_i) \cdot m\nabla p. \quad (44)$$

By Cauchy–Schwarz inequality, it follows that

$$\frac{1}{\alpha} \int_{\partial\omega} q^2 + \frac{1}{2} \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} \|m\nabla p\|^2 \leq c_1 - \int_{\omega} b\nabla t \cdot \nabla q. \quad (45)$$

Here and in the sequel,  $c_i$  denotes various positive numbers independent of  $\eta$ ,  $p$  and  $q$ . Setting  $\psi = q$  in (43) and using  $m \leq c_2 a$  and Cauchy–Schwarz inequality, we get

$$\frac{1}{2} \int_{\omega} a \|\nabla q\|^2 + \eta \int_{\partial\omega} q^2 \leq c_3 \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} \|m\nabla p\|^2 + c_4.$$

Dividing by  $2c_3$ , adding to (45) and using  $b \leq c_5 a$  and Cauchy–Schwarz inequality, we get

$$\frac{1}{\alpha} \int_{\partial\omega} q^2 + c_6 \int_{\omega} a \|\nabla q\|^2 \leq c_7. \quad (46)$$

This implies that  $q$  is bounded in  $H^1(\omega)$ .

Now, let us bound  $p$ . Setting  $\varphi = p$  in (42) and using  $b \leq c_5 a$ , Cauchy–Schwarz inequality and (46), we get

$$\frac{1}{2} \int_{\omega} a \|\nabla p\|^2 + \eta \int_{\partial\omega} p^2 \leq c_8 - \frac{1}{\alpha} \int_{\partial\omega} pq \leq c_8 + c_9 \left( \int_{\partial\omega} p^2 \right)^{1/2}. \quad (47)$$

Setting  $\psi = 1$  in (43), we get  $\int_{\partial\omega} q = 0$ , and then, setting  $\varphi = 1$  in (42), we get

$$\int_{\partial\omega} p = -\frac{1}{\alpha\eta} \int_{\partial\omega} q = 0. \quad (48)$$

Therefore Poincaré–Wirtinger inequality  $(\int_{\partial\omega} p^2)^{1/2} \leq c_{10}(\int_{\omega} \|\nabla p\|^2)^{1/2}$  holds. Then (47) gives  $(\int_{\omega} \|\nabla p\|^2)^{1/2} \leq c_{11}$  and  $p$  is bounded in  $H^1(\omega)$ .

Let now  $\eta_n \rightarrow 0$ . There exists a subsequence such that the corresponding solution  $(p_n, q_n)$  of (42)–(43) goes to some limit  $(p, q)$  weakly in  $H^1(\omega) \times H^1(\omega)$  and therefore strongly in  $L^2(\omega) \times L^2(\omega)$  and its trace converges strongly in  $L^2(\partial\omega) \times L^2(\partial\omega)$ . Then the limit  $(p, q)$  satisfies exact equations (40)–(41). These equations possessing a unique solution, as seen in § 3.4, all subsequences go to the same limit and therefore the whole sequence  $(p_n, q_n)$  goes to  $(p, q)$ .

#### 4.3. Computation of wind

Practically, we compute the unique solution  $(p, q)$  of (approximated) coupled equations (42)–(43) for a small parameter  $\eta$ , and then we compute the (approximated adjusted) wind velocity  $V(x, z)$  at every point  $(x, z)$  in terms of  $\nabla p(x)$  using expression (16).

#### 4.4. Finite element approximation

Let us discretize the approximated equations (42)–(43). Let  $\mathcal{T}_H$  be a uniform triangulation of  $\omega$  corresponding to a discretization parameter  $H$  and let  $V_H$  be the associated space of  $P_1$  (or  $P_2$ ) finite elements. Besides a better order of convergence, a reason in favor of  $P_2$  against  $P_1$  is that in practical applications, the variable of physical interest is the wind velocity  $V$  which is obtained from the potential  $p$  using expression (16), involving derivatives.

Choosing a finite element basis  $\{\phi_i\}$  for  $V_H$ , we introduce the following matrices

$$G = \left\{ \int_{\omega} a \nabla \phi_r \cdot \nabla \phi_k + \eta \int_{\partial\omega} \phi_r \phi_k \right\}_{r,k},$$

$$C_1 = \left\{ \frac{1}{\alpha} \int_{\partial\omega} \phi_r \phi_k \right\}_{r,k}, \quad C_2 = \left\{ \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} m^2 \nabla \phi_r \cdot \nabla \phi_k \right\}_{r,k}$$

and the vectors

$$f_p = \left\{ - \int_{\omega} b \nabla t \cdot \nabla \phi_r \right\}_r, \quad f_q = \left\{ - \sum_{i=1}^N \int_{\omega} \rho_{\epsilon,i} (n \nabla t - V_i) m \cdot \nabla \phi_r \right\}_r.$$

Then, the discrete problem associated to (42)–(43) is the following linear algebraic system:

$$\begin{bmatrix} G & C_1 \\ -C_2 & G \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} f_p \\ f_q \end{bmatrix} \quad (49)$$

The matrix in (49) being nonsymmetric and very ill-conditioned, most of the standard iterative methods fail to converge or have a very slow convergence (this is the case of GMRES-ILU preconditioned). For moderate number of unknowns we use the state-of-the-art sparse LU factorization [4]. In [5] a highly effective solution method is obtained by means of a preconditioned Schur complement approximate, leading to a nonsymmetric system that can be solved by GMRES in a constant number of iterations. For the description and a complete numerical analysis of this approximate see [5]. According to this numerical analysis and to the numerical experiments described in the following section, the number of iterations appears to be insensitive to  $\eta$  and only mildly dependent on  $\alpha$ . Note that  $p$  is determined up to a constant, but only  $\nabla p$  is needed to compute the wind velocity  $V$ . Then we choose  $\eta$  small enough so that the perturbation term does not affect the value of  $\nabla p$  up to the desired precision.

## 5. NUMERICAL EXAMPLES

*5.1. Example 1: Effect of a topography and of a temperature gradient*

In this section we consider the effect of two hills on the wind, as well as the effect of the temperature gradient in a square of 6 by 6 kilometers. The ground height and ground temperature are shown on Figure 1.

Figure 1. Ground height and ground temperature (Example 1)

The wind velocity is supposed to be known (by experimental measurements) at four points of horizontal coordinates  $x = (1., 1.), (5., 1.), (5., 5.), (1., 5.)$  and of height  $z = 0.1 + h(x)$ , with the same value  $V(x, z) = (2., 0.)$  and we take  $\alpha = 0.001$ .

Figure 2 shows the computed adjoint state and potential, and Figure 3 shows the computed velocity module and wind field on the ground surface, that is for  $z = h(x)$ . As expected, the wind is deflected by hills and it converges in the hot region.

Figure 2. Adjoint state and potential (Example 1)

Figure 3. Velocity module and wind field (Example 1)

### *5.2. Real data for Example 2*

We have considered a simulation using realistic wind data that have been supplied by *Desarrollos Eólicos S.A., (DESA)* in several measurement points for an episode along March

21, 2003, see [6].

The studied three-dimensional domain  $\Omega$  is located near Lugo, Spain, at  $43N$  of latitude and it is horizontally limited by four points of UTM coordinates (609980, 4799020), (626000,4799020), (626000, 4813040) and (609980, 4813040), respectively. The upper boundary  $A$  of  $\Omega$  has been taken at a height  $\delta = 1080$  m. A digital elevation map was provided by *DESA* on a quadrilateral grid of element size  $20 \times 20$  m. The  $X$  axis corresponds to East direction and the  $Y$  one to North. Thus, we are working with a region of  $16020 \times 14020$  m. The minimum and maximum terrain heights are 420 m and 1020 m, respectively. Figure 4 represents a color map of the heights of the terrain.

Wind has been measured every 10 minute at 5 stations which are plotted on Figure 4: from North to South, we find E208, E212, E242, E206 and E283. At stations E208 and E212, the wind was measured at two different heights. Their coordinates are given in Table I.

| Station | UTM-E  | UTM-N   | Height |
|---------|--------|---------|--------|
| E206    | 615396 | 4805218 | 924.8  |
| E208    | 616917 | 4807256 | 945.0  |
| E212    | 617423 | 4806382 | 895.0  |
| E242    | 618290 | 4806136 | 873.2  |
| E283    | 617473 | 4804111 | 849.0  |

Table I. Coordinates of stations (in meters)

Roughness is an essential factor on the characteristics of the resulting wind profile. In this case, the roughness length values are 0.03 m, 0.05 m, 0.08 m, 0.3 m and 0.8 m.

Figure 4. Terrain heights and stations: From North to South E208, E212, E242, E206 and E283

### 5.3. Example 2: Wind computation for above real data

We proceed in three steps:

- First, we estimate the main parameters of our optimal control wind model.
- We compute the wind with our model using data of *input stations* E206, E208 and E212.
- Finally, we compare the computed wind to the measured wind at *control stations* E242 and E283 (Table II). They are also compared at input stations (Table III) but this is less significant since, there, the computed wind is optimized to be close to data.

We propose a quadratic adjustment of the friction coefficient in terms of the roughness of the terrain, i.e.,  $\zeta = a_0 + a_1 z_0 + a_2 z_0^2$ . We have used a standard genetic algorithm code (*pgapack* library), with string real coding, based on the model developed by Levine [7] to look for optimal values of these quadratic adjustment parameters  $a_0$ ,  $a_1$  and  $a_2$ . We search for the optimum of the linear parameter  $a_0$  in  $[1, 10]$ , the first order parameter  $a_1$  in  $[0, 5]$  and the second order parameter  $a_2$  in  $[-0.05, 0.05]$ . The values of the coefficients  $a_i$  depends slightly of the meteorological conditions, which means that the friction coefficient depends slightly on the solution. Typical values of the coefficients are  $a_0 = 3$ ,  $a_1 = 0.5$  and  $a_2 = -0.01$  showing in practice a linear dependence of the friction coefficient with the roughness.

We have computed the wind every 10 minutes throughout the day with our optimal control model for the above adjusted parameters, using wind data only at stations E206, E208 and E212 (*input stations*). The data at the two other stations, E242 and E283 (*control stations*), are not used as input; they are used to control the efficiency of our model. The measured and the computed wind velocities at these control stations are shown in Figures 5 and 6. The estimation of the parameters  $a_i$  has been carried out only once each hour (24 computations along the day) using the first experimental wind measurements that are available at the beginning of each hour.

In order to compare our model (called 2.5-D model) with classical wind adjustment models, these figures include the wind velocities computed by the three-dimensional mass consistent model (3-D model) described in [6].

Errors of the computed winds with respect to the measured winds at control stations are given in Table II. Remark that the present 2.5-D model provides better results at the control

Figure 5. Comparison of measured and computed wind velocities at control station E242

Figure 6. Comparison of measured and computed wind velocities at control station E283

station E242 which is close to the input stations (see Figure 4). On the contrary, the 3-D model is more accurate at the control station E283 which is far from the input stations. This error can be explained by the fact that the hypothesis of the 2.5-D model are broken by the more rugged terrain at the station E283 (we can see on Figure 4 that the level set lines are stretched close to E283).

| Stations and<br>control points | Average<br>measured<br>wind | Average<br>computed<br>wind | %<br>average<br>error | Maximum<br>absolute<br>error | Minimum<br>absolute<br>error | Model |
|--------------------------------|-----------------------------|-----------------------------|-----------------------|------------------------------|------------------------------|-------|
| E242 (40 <i>m</i> )            | 8.40                        | 10.69                       | 27.24%                | 5.09                         | 0.09                         | 3-D   |
|                                | 8.40                        | 8.49                        | 1.05 %                | 2.54                         | 0.01                         | 2.5-D |
| E283 (49 <i>m</i> )            | 13.62                       | 12.95                       | 4.94 %                | 3.04                         | 0.02                         | 3-D   |
|                                | 13.62                       | 7.73                        | 43.28 %               | 9.88                         | 1.34                         | 2.5-D |

Table II. Error at control stations

Errors at input stations are given in Table III. As expected, average errors are small.

## 6. CONCLUSIONS

We have presented a physical model for wind field adjustment which derives from an asymptotic approximation of Navier-Stokes equations. It provides a detailed extrapolation of punctual wind data preserving air incompressibility and orography effects and incorporating thermal effects if required.

A comparison of the results with real data shows that this model may be interesting for local wind adjustment: it gives very good results close enough to the measurements points, although nonlinear effects are not taken into account.

From the computational point of view, in front of other mass consistent models, it has the advantage to provide a three dimensional wind field performing only two dimensional linear computations.

| Stations and<br>control points | Average<br>measured<br>wind | Average<br>computed<br>wind | %<br>average<br>error | Maximum<br>absolute<br>error | Minimum<br>absolute<br>error | Model |
|--------------------------------|-----------------------------|-----------------------------|-----------------------|------------------------------|------------------------------|-------|
| E206 (49 m)                    | 15.37                       | 15.50                       | 0.81 %                | 0.46                         | 0.01                         | 3-D   |
|                                | 15.37                       | 14.82                       | 3.62 %                | 1.15                         | 0.23                         | 2.5-D |
| E208 (15 m)                    | 8.57                        | 8.98                        | 4.74 %                | 1.25                         | 0.00                         | 3-D   |
|                                | 8.57                        | 9.13                        | 6.52 %                | 2.45                         | 0.01                         | 2.5-D |
| E208 (30 m)                    | 9.25                        | 9.92                        | 7.21 %                | 1.36                         | 0.05                         | 3-D   |
|                                | 9.25                        | 9.42                        | 1.82 %                | 1.41                         | 0.00                         | 2.5-D |
| E212 (15 m)                    | 8.46                        | 8.44                        | 0.20 %                | 0.63                         | 0.00                         | 3-D   |
|                                | 8.46                        | 8.18                        | 3.33 %                | 1.89                         | 0.01                         | 2.5-D |
| E212 (30 m)                    | 9.02                        | 9.85                        | 9.25 %                | 1.60                         | 0.31                         | 3-D   |
|                                | 9.02                        | 8.46                        | 6.17 %                | 2.16                         | 0.01                         | 2.5-D |

Table III. Error at input stations

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