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NUMERICAL SIMULATION OF SHALLOW LAKES AND SEAS: FURTHER RESULTS

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Abstract. *The motion of a fluid subject to Navier–Stokes equations with Coriolis force, to a traction condition at the surface and to a friction condition at the bottom is investigated. An asymptotic model is derived as the aspect ratio δ = depth/width of the domain goes to 0. The 3D limit velocity is given in terms of wind, bathymetry, depth and of a 2D potential. Numerical simulation is carried out on North Pacific.*

1 INTRODUCTION

In previous papers[4,5,6], asymptotic models for shallow lakes and seas were derived assuming that wind traction is prescribed and that the fluid is at rest at the bottom. A slipping condition without friction at the bottom was also considered[7]. Here, the more general case of a fluid subject to a traction condition at the surface and to a friction condition at the bottom is investigated. The friction condition considered here is used in limnology or in coastal models to take into account the boundary layer at the bottom[3,15,16].

In view of the small aspect ratio δ = depth/width of most lakes and seas, an asymptotic analysis as $\delta \to 0$ is performed. It leads to a limit momentum equation driven by the vertical diffusion and by the part of Coriolis force due to the East-West component of earth rotation, to the hydrostatic approximation and to the incompressibility equation. The limit traction and friction condition drive the vertical derivative of the horizontal velocity.

When Reynolds number is not too large, convection terms are neglectible. Then, the limit potential is driven by a 2D equation and the 3D limit velocity is given in terms of wind, bathymetry, depth and potential. This provides a 3D velocity for the single cost of a 2D computation which is carried out for Pacific ocean.

Various evolution models including possibly thermics and salinity were used[12,23]. Let us give a limited list of references. *Shallow water* 2D equations are frequently used[1,20,22,26]. Mathematical analysis of *Homogeneous wind driven* 2D model was performed[10,11] as well as mathematical and numerical analysis of 3D *Primitive equations*[8,9,14,18,19,25]. Other models for lakes or ocean were discussed[2,10,12,13,18,27].

2 ADIMENSIONALIZED STATIONARY NAVIER-STOKES EQUATIONS

Let us consider a fluid occupying the following dimensionless domain of $\mathbb{R}^3$

$$d = \{(x, z) : x \in \mathcal{O},\ -h(x) < z < 0\}$$

where the horizontal section $\mathcal{O}$ is an open subset of $\mathbb{R}^2$ and the depth h is a continuous function on $\mathcal{O}$ vanishing on $\partial\mathcal{O}$. The boundary is $\partial d = b \cup s \cup \partial s$ where the bottom b, the surface s and the shore ∂s are respectively:

$$b = \{(x, -h(x)) : x \in \mathcal{O}\}, \quad s = \{(x, 0) : x \in \mathcal{O}\}, \quad \partial s = \partial b = \{(x, 0) : x \in \partial\mathcal{O}\}.$$

The adimensionalized velocity and potential (u, p) satisfy[23,7]

$$\frac{1}{\mathrm{Re}}(-\Delta u + \nabla p) + \frac{1}{\mathrm{Ro}}\varpi \times u + u \cdot \nabla u = 0, \quad \nabla \cdot u = 0, \quad \text{in } d. \tag{1}$$

Here Re and Ro are Reynolds and Rossby numbers, ϖ is the earth rotation unit-vector and $\times$ is the $3D$ vector product.

The vertical component of the velocity is distinguished from the horizontal one since it plays a different part, on one hand because the surface is horizontal and on the other hand because the depth is small in front of the width. More precisely, we denote $u = (v, w)$ where $v = (v_1, v_2)$ is the horizontal velocity. Similarly, the vertical coordinate z is distinguished from the horizontal ones $x = (x_1, x_2)$ and the gradient is splitted into $\nabla = (\nabla_x, \partial_z) = (\partial_{x_1}, \partial_{x_2}, \partial_z)$. Then, the wind traction condition and the surface invariableness read

$$\delta \partial_z v = |v_{\text{air}}| \left(v_{\text{air}} + \delta \theta_s \left(v_{\text{air}} - v\right)\right), \quad w = 0, \quad \text{on } s \tag{2}$$

where v_{air} is the adimensional air velocity and θ_b is a drag coefficient. The slipping with friction at the bottom reads

$$\frac{\partial u}{\partial n}\Big|_{\text{tang}} = -\theta_b \, u, \quad u \cdot n = 0, \quad \text{on } b$$

where θ_b is a drag coefficient modelling bottom rugosity. The lack of horizontal flux through the shore reads

$$\overline{v} \cdot n_{\partial \mathcal{O}} = 0 \tag{3}$$

where $\overline{v}(x) = \int_{-h(x)}^{0} v(x, z)\, dz$ and $n_{\partial \mathcal{O}}$ is the normal vector to $\partial \mathcal{O}$ in the horizontal plane. Remark that $\nabla \cdot u = 0$ and $u \cdot n = 0$ on $b \cup s$ imply $\nabla_x \cdot \overline{v} = 0$ on $\partial \mathcal{O}$.

3 ASYMPTOTIC MODEL

Let us now simplify the equations governing the fluid by only retaining the leading terms as the aspect ratio δ = depth/width is small. To avoid confusion, the solution of equations containing only the leading terms is denoted $(\mathbf{v}, \mathbf{w}, \mathbf{p})$. It is an approximation of (v, w, p).

1 The general model

If

$$\delta \ll 1, \quad \delta^2 \text{Re} \approx \gamma, \quad \frac{\delta^2 \text{Re}}{\text{Ro}} \approx e, \quad \delta \theta_s \approx \tau_s, \quad \delta \theta_b \approx \tau_b, \tag{4}$$

then the approached velocity $(\mathbf{v}, \mathbf{w})$ and potential $\mathbf{p}$ satisfy

$$-\delta^2 \partial_z^2 \mathbf{v} + k \mathbf{v}^{\perp} + \gamma \left(\mathbf{v} \cdot \nabla_x \mathbf{v} + \mathbf{w} \partial_z \mathbf{v}\right) + \nabla_x \mathbf{p} = 0, \quad \partial_z \mathbf{p} = 0, \quad \nabla_x \cdot \mathbf{v} + \partial_z \mathbf{w} = 0 \tag{5}$$

in the domain d, where $k = e\, \overline{\omega}_3$ and $\mathbf{v}^{\perp} = (-\mathbf{v}_2, \mathbf{v}_1)$, and

$$\begin{cases} \delta \partial_z \mathbf{v} = |v_{\text{air}}| \left(v_{\text{air}} + \tau_s (v_{\text{air}} - \mathbf{v})\right), \quad \mathbf{w} = 0, & \text{on the surface } s; \\ \delta \partial_z \mathbf{v} = \tau_b \, \mathbf{v}, \quad (\mathbf{v}, \mathbf{w}) \cdot n = 0, & \text{on the bottom } b; \\ \overline{\mathbf{v}} \cdot n_{\partial \mathcal{O}} = 0, & \text{on the shore } \partial \mathcal{O}. \end{cases} \tag{6}$$

Remark that (5) and (6) imply
$$\nabla_x \cdot \overline{\mathbf{v}} = 0. \tag{7}$$

2 The vertical-geostrophic model

Without the convection term, that is if $\gamma = 0$, Equations (5) reduce to the *vertical-geostrophic* equations to which the sequel is devoted.

3 Other particular cases

Without Coriolis term, that is if $k = 0$, Equations (5) are similar to the so-called *Prandtl* equations, which govern the boundary layer in a laminar flow outside a body, see Equations (39,5)–(39,6) p. 226 in[17].

Without the diffusion and convection terms, that is if $\delta = \gamma = 0$, Equations (5) are closed to the so-called *geostrophic* approximation. It reduces to $\mathbf{p} = \mathbf{p}(x)$, $\mathbf{v} = -\nabla_x^\perp \mathbf{p}/k$ and $\mathbf{w} = z\nabla_x^\perp \mathbf{p}/k$ thus (7) and $\overline{\mathbf{v}} \cdot n_{\partial\mathcal{O}} = 0$, see (6), give

$$\nabla_x \cdot \Big(\frac{h}{k}\,\nabla_x^\perp \mathbf{p}\Big) = 0, \quad \Big(\frac{h}{k}\,\nabla_x^\perp \mathbf{p}\Big) \cdot n_{\partial\mathcal{O}} = 0.$$

The other boundary conditions in (6) cannot then be satisfied. Should convergence be proved, it would only be local with boundary layers.

Without the diffusion and Coriolis terms, that is if $\delta = k = 0$, Equations (5) correspond to the stationary *Euler* equations degenerated in the vertical direction. Their resolution is an open problem for which the boundary conditions (6) seems to be too strong.

4 MATHEMATICAL JUSTIFICATION

A convergence result of conveniently scaled solutions in a reference domain D is given here, for $\gamma = 0$. It justifies the above derivation of the vertical-geostrophic model.

Using the scaling
$$z = \delta Z, \quad h(x) = \delta H(x), \tag{8}$$

the varying domain d is mapped onto the following reference domain of $\mathbb{R}^3$:

$$D = \{(x, Z) : x \in \mathcal{O},\ -H(x) < Z < 0\}.$$

To any velocity and potential (v, w, p) in d, scaled velocity and potential (v_D, w_D, p_D) in D are associated by:

$$v_D(x, Z) = v(x, z), \quad w_D(x, Z) = \frac{1}{\delta}\, w(x, z), \quad p_D(x, Z) = p(x, z). \tag{9}$$

The tridimensionnal domain d is assumed to be such that

$$\begin{cases} \mathcal{O} \text{ is a connected open bounded subset of } \mathbb{R}^2, \\ \partial\mathcal{O} \text{ is locally the graph of a Lipschitz continuous function,} \\ h \in \mathcal{C}^1(\overline{\mathcal{O}}),\ h > 0 \text{ in } \mathcal{O},\ h = 0 \text{ on } \partial\mathcal{O},\ \nabla h \cdot n_{\partial\mathcal{O}} > 0 \text{ on } \partial\mathcal{O}. \end{cases} \tag{10}$$

Air velocity and friction coefficients are assumed to satisfy

$$v_{\text{air}} \in (L^\infty(\mathcal{O}))^2, \quad \tau_s \geq 0, \quad \tau_b > 0. \tag{11}$$

Finally the coefficients are assumed to satisfy

$$\delta \to 0, \quad \delta\text{Re} \to 0, \quad \frac{\delta^2\text{Re}}{\text{Ro}} \to e, \quad \delta\theta_s \to \tau_s, \quad \delta\theta_b \to \tau_b. \tag{12}$$

Then the following convergence result holds (the proof will be published later).

Theorem 1. *There exists a solution* (v, w, p) *of stationary Navier–Stokes equations* (1) *with boundary conditions* (2)–(3) *whose image by the scaling* (8)–(9) *goes to the image of a solution* $(\mathbf{v}, \mathbf{w}, \mathbf{p})$ *of the vertical-geostrophic equations* (5)–(6) *in the following sense: for all* $r < 3/2$,

$$(v_D, w_D, p_D) \to (\mathbf{v}_D, \mathbf{w}_D, \mathbf{p}_D) \text{ in } (L^2(D))^2 \times H^{-1}(D) \times L^r(D). \ \square$$

Formally, see (4), the convection terms are neglectible as soon as $\delta^2\text{Re} \to 0$. In (12), the stronger assumption $\delta\text{Re} \to 0$ is stated. This restriction is purely mathematical. It would be interesting to fill this gap between formal analysis and mathematical proof, that is to prove convergence assuming only $\delta^2\text{Re} \to 0$. It would be still more interesting to prove convergence for $\gamma \neq 0$.

5 REDUCTION TO A 2D MODEL

Here Equations (5)–(6) are reduced to a 2D model on $\mathbf{p} = \mathbf{p}(x)$ on the surface when $\gamma = 0$. The velocity is expressed in terms of z, x and $\nabla_x \mathbf{p}(x)$.

4 Computation of the velocity

The hydrostatic condition $\partial_z \mathbf{p} = 0$ gives $\mathbf{p} = \mathbf{p}(x)$. Let us fix x and solve the equations with respect to z. The momentum equation $-\delta^2\partial_z^2\mathbf{v} + k\mathbf{v}^\perp + \nabla_x\mathbf{p} = 0$, the traction condition $\delta\partial_z\mathbf{v}(0) = |v_{\text{air}}|\left(v_{\text{air}} + \tau_s(v_{\text{air}} - \mathbf{v}(0))\right)$ and the friction condition $\delta\partial_z\mathbf{v}(-h) = \tau_b\mathbf{v}(-h)$ give

$$\mathbf{v} = (a_{\mathbf{v}}I + b_{\mathbf{v}}\varsigma A)\nabla_x\mathbf{p} + (c_{\mathbf{v}}I + d_{\mathbf{v}}\varsigma A)f \tag{13}$$

where $a_{\mathbf{v}}$, $b_{\mathbf{v}}$, $c_{\mathbf{v}}$ and $d_{\mathbf{v}}$ are functions of x and z satisfying

$$\begin{cases} -\delta^2\partial_z^2 a_{\mathbf{v}} - |k| b_{\mathbf{v}} + 1 = -\delta\partial_z^2 b_{\mathbf{v}} + |k| a_{\mathbf{v}} = 0, \\ -\delta^2\partial_z^2 c_{\mathbf{v}} - |k| d_{\mathbf{v}} = -\delta^2\partial_z^2 d_{\mathbf{v}} + |k| c_{\mathbf{v}} = 0, \\ \delta\partial_z a_{\mathbf{v}}(-h) - \tau_b a_{\mathbf{v}}(-h) = \delta\partial_z b_{\mathbf{v}}(-h) - \tau_b b_{\mathbf{v}}(-h) = 0, \\ \delta\partial_z c_{\mathbf{v}}(-h) - \tau_b c_{\mathbf{v}}(-h) = \delta\partial_z d_{\mathbf{v}}(-h) - \tau_b d_{\mathbf{v}}(-h) = 0, \\ \delta\partial_z a_{\mathbf{v}}(0) + \tau_a a_{\mathbf{v}}(0) = \delta\partial_z b_{\mathbf{v}}(0) + \tau_a b_{\mathbf{v}}(0) = \delta\partial_z d_{\mathbf{v}}(0) + \tau_a d_{\mathbf{v}}(0) = 0, \\ \delta\partial_z c_{\mathbf{v}}(0) + \tau_a c_{\mathbf{v}}(0) = 1, \end{cases}$$

where $Af = f^\perp = (-f_2, f_1)$ and where ς, τ_a and f are functions of x defined by

$$\varsigma = \text{ sign of } k, \quad \tau_a = \tau_s\,|v_{\text{air}}|, \quad f = (1+\tau_s)\,|v_{\text{air}}| v_{\text{air}}. \tag{14}$$

If $k(x) \neq 0$, the unique solution of these equations is

$$\begin{aligned} a_{\mathbf{v}} &= \alpha_1 \sinh(-\kappa Z)\sin(-\kappa Z) + \beta_1 \cosh(-\kappa Z)\cos(-\kappa Z) \\ &\quad + \gamma_1 \sinh(-\kappa Z)\cos(-\kappa Z) + \delta_1 \cosh(-\kappa Z)\sin(-\kappa Z), \\ b_{\mathbf{v}} &= \frac{1}{2\kappa^2} - \alpha_1 \cosh(-\kappa Z)\cos(-\kappa Z) + \beta_1 \sinh(-\kappa Z)\sin(-\kappa Z) \\ &\quad + \gamma_1 \cosh(-\kappa Z)\sin(-\kappa Z) - \delta_1 \sinh(-\kappa Z)\cos(-\kappa Z), \\ c_{\mathbf{v}} &= \alpha_2 \sinh(-\kappa Z)\sin(-\kappa Z) + \beta_2 \cosh(-\kappa Z)\cos(-\kappa Z) \\ &\quad + \gamma_2 \sinh(-\kappa Z)\cos(-\kappa Z) + \delta_2 \cosh(-\kappa Z)\sin(-\kappa Z), \\ d_{\mathbf{v}} &= -\alpha_2 \cosh(-\kappa Z)\cos(-\kappa Z) + \beta_2 \sinh(-\kappa Z)\sin(-\kappa Z) \\ &\quad + \gamma_2 \cosh(-\kappa Z)\sin(-\kappa Z) - \delta_2 \sinh(-\kappa Z)\cos(-\kappa Z), \end{aligned}$$

where

$$Z = \frac{z}{\delta}, \quad \kappa(x) = \sqrt{\frac{|k(x)|}{2}}.$$

The coefficients $\alpha_1, \ldots, \delta_1, \alpha_2, \ldots, \delta_2$ are functions of x defined by

$$\alpha_1 = \frac{AC_1 - BD_1}{2\kappa^2 M}, \quad \beta_1 = \frac{AD_1 + BC_1}{2\kappa^2 M}, \quad \alpha_2 = \frac{AC_2 - BD_2}{M}, \quad \beta_2 = \frac{AD_2 + BC_2}{M},$$

$$\gamma_1 = \frac{\tau_a}{2\kappa}\Big(\frac{1}{2\kappa^2} - \alpha_1 + \beta_1\Big), \quad \delta_1 = \frac{\tau_a}{2\kappa}\Big(-\frac{1}{2\kappa^2} + \alpha_1 + \beta_1\Big),$$

$$\gamma_2 = \frac{1}{2\kappa}(-1 + \tau_a(-\alpha_2 + \beta_2)), \quad \delta_2 = \frac{1}{2\kappa}(-1 + \tau_a(\alpha_2 + \beta_2)),$$

where

$$\begin{gathered} A = (2\kappa^2 + \tau_a\tau_b)Cs + 2\kappa(\tau_b + \tau_a)Ss + (2\kappa^2 - \tau_a\tau_b)Sc, \\ B = (-2\kappa^2 + \tau_a\tau_b)Cs + 2\kappa(\tau_b + \tau_a)Cc + (2\kappa^2 + \tau_a\tau_b)Sc, \\ C_1 = \tau_a(2\kappa Ss + \tau_b(-Sc + Cs)), \quad D_1 = -2\kappa\tau_b - \tau_a(2\kappa Cc + \tau_b(Sc + Cs)), \\ C_2 = 2\kappa Cc + \tau_b(Sc + Cs), \quad D_2 = 2\kappa Ss + \tau_b(-Sc + Cs), \quad M = A^2 + B^2, \\ S = \sinh\Big(\frac{\kappa h}{\delta}\Big), \quad C = \cosh\Big(\frac{\kappa h}{\delta}\Big), \quad s = \sin\Big(\frac{\kappa h}{\delta}\Big), \quad c = \cos\Big(\frac{\kappa h}{\delta}\Big). \end{gathered}$$

The equation $\nabla \cdot \mathbf{v} + \partial_z \mathbf{w} = 0$ and $\mathbf{w}(0) = 0$ give $\mathbf{w} = \int_z^0 \nabla_x \cdot \mathbf{v} = \nabla_x \cdot (\int_z^0 \mathbf{v})$. Therefore

$$\mathbf{w} = \delta \nabla_x \cdot ((a_{\mathbf{w}} I + b_{\mathbf{w}} \varsigma A) \nabla_x \mathbf{p} + (c_{\mathbf{w}} I + d_{\mathbf{w}} \varsigma A) f) \tag{15}$$

where

$$a_{\mathbf{w}} = \frac{1}{\delta} \int_z^0 a_{\mathbf{v}}, \quad b_{\mathbf{w}} = \frac{1}{\delta} \int_z^0 b_{\mathbf{v}}, \quad c_{\mathbf{w}} = \frac{1}{\delta} \int_z^0 c_{\mathbf{v}}, \quad d_{\mathbf{w}} = \frac{1}{\delta} \int_z^0 d_{\mathbf{v}}.$$

We did put a multiplicative factor δ in (15) because it is the magnitude of $\mathbf{w}$. This provides a magnitude one for the coefficients $a_{\mathbf{w}}, \ldots, d_{\mathbf{w}}$. They are

$$\begin{aligned}
a_{\mathbf{w}} = {} & \frac{\delta_1 - \gamma_1}{2\kappa} - \frac{\alpha_1}{2\kappa}\Big(\sinh(-\kappa Z)\cos(-\kappa Z) - \cosh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& + \frac{\beta_1}{2\kappa}\Big(\sinh(-\kappa Z)\cos(-\kappa Z) + \cosh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& + \frac{\gamma_1}{2\kappa}\Big(\sinh(-\kappa Z)\sin(-\kappa Z) + \cosh(-\kappa Z)\cos(-\kappa Z)\Big) \\
& - \frac{\delta_1}{2\kappa}\Big(\cosh(-\kappa Z)\cos(-\kappa Z) - \sinh(-\kappa Z)\sin(-\kappa Z)\Big), \\
b_{\mathbf{w}} = {} & -\frac{Z}{2\kappa^2} + \frac{\gamma_1 + \delta_1}{2\kappa} - \frac{\alpha_1}{2\kappa}\Big(\sinh(-\kappa Z)\cos(-\kappa Z) + \cosh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& - \frac{\beta_1}{2\kappa}\Big(\sinh(-\kappa Z)\cos(-\kappa Z) - \cosh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& - \frac{\gamma_1}{2\kappa}\Big(\cosh(-\kappa Z)\cos(-\kappa Z) - \sinh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& - \frac{\delta_1}{2\kappa}\Big(\sinh(-\kappa Z)\sin(-\kappa Z) + \cosh(-\kappa Z)\cos(-\kappa Z)\Big), \\
c_{\mathbf{w}} = {} & \frac{\delta_2 - \gamma_2}{2\kappa} - \frac{\alpha_2}{2\kappa}\Big(\sinh(-\kappa Z)\cos(-\kappa Z) - \cosh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& + \frac{\beta_2}{2\kappa}\Big(\sinh(-\kappa Z)\cos(-\kappa Z) + \cosh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& + \frac{\gamma_2}{2\kappa}\Big(\sinh(-\kappa Z)\sin(-\kappa Z) + \cosh(-\kappa Z)\cos(-\kappa Z)\Big) \\
& - \frac{\delta_2}{2\kappa}\Big(\cosh(-\kappa Z)\cos(-\kappa Z) - \sinh(-\kappa Z)\sin(-\kappa Z)\Big), \\
d_{\mathbf{w}} = {} & \frac{\delta_2 + \gamma_2}{2\kappa} - \frac{\alpha_2}{2\kappa}\Big(\sinh(-\kappa Z)\cos(-\kappa Z) + \cosh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& - \frac{\beta_2}{2\kappa}\Big(\sinh(-\kappa Z)\cos(-\kappa Z) - \cosh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& - \frac{\gamma_2}{2\kappa}\Big(\cosh(-\kappa Z)\cos(-\kappa Z) - \sinh(-\kappa Z)\sin(-\kappa Z)\Big) \\
& - \frac{\delta_2}{2\kappa}\Big(\sinh(-\kappa Z)\sin(-\kappa Z) + \cosh(-\kappa Z)\cos(-\kappa Z)\Big).
\end{aligned}$$

5 Potential equation

From (13),

$$\overline{\mathbf{v}} = \int_{-h}^{0} \mathbf{v} = -((a_{\mathbf{p}} I + b_{\mathbf{p}} \varsigma A)\nabla_x \mathbf{p} + (c_{\mathbf{p}} I + d_{\mathbf{p}} \varsigma A) f) \tag{16}$$

where

$$a_{\mathbf{p}} = -a_{\mathbf{w}}(-h), \quad b_{\mathbf{p}} = -b_{\mathbf{w}}(-h), \quad c_{\mathbf{p}} = -c_{\mathbf{w}}(-h), \quad d_{\mathbf{p}} = -d_{\mathbf{w}}(-h).$$

Then (7) and the last equation in (6) yield the following 2D problem, in $\mathcal{O}$:

$$\nabla_x \cdot ((a_{\mathbf{p}} I + b_{\mathbf{p}} \varsigma A)\nabla_x \mathbf{p} + (c_{\mathbf{p}} I + d_{\mathbf{p}} \varsigma A) f) = 0, \tag{17}$$

$$((a_{\mathbf{p}} I + b_{\mathbf{p}} \varsigma A)\nabla_x \mathbf{p} + (c_{\mathbf{p}} I + d_{\mathbf{p}} \varsigma A) f) \cdot n_{\partial\mathcal{O}} = 0. \tag{18}$$

6 Miscelleanous

The above expressions of $a_{\mathbf{v}}, b_{\mathbf{v}}, \ldots, d_{\mathbf{p}}$ are not valid at points where $k(x) = 0$, that is at equator, since then $\kappa(x) = 0$. However these coefficients possess continuous extensions at such points.

Expressions for a prescribed traction at the surface and adherence to the bottom[6] are found again by setting $\tau_s = 0$ and letting $\tau_b \to +\infty$.

6 RESOLUTION OF THE 2D MODEL

Let us endow the weighted space

$$\boldsymbol{A}(\mathcal{O}) = \{\mathbf{p} \in L^2_{\mathrm{loc}}(\mathcal{O}) : h\nabla_x \mathbf{p} \in (L^2(\mathcal{O}))^2, \int_{\mathcal{O}} h\mathbf{p}\, dx = 0\}$$

with the norm $\|\mathbf{p}\|_{\boldsymbol{A}(\mathcal{O})} = (\int_{\mathcal{O}} |h\nabla_x \mathbf{p}|^2)^{1/2}$. Notice that the weight h vanishes on the boundary $\partial\mathcal{O}$, but is non negative in $\mathcal{O}$, therefore $\boldsymbol{A}(\mathcal{O}) \subset H^1_{\mathrm{loc}}(\mathcal{O})$. Let us solve Equations (17)–(18) in this space.

Proposition 2. *Let* (10) *and* (11) *be satisfied. There exists a unique solution* $\mathbf{p} \in \boldsymbol{A}(\mathcal{O})$ *of the following variational equation: for all* $\Xi \in \boldsymbol{A}(\mathcal{O})$,

$$\int_{\mathcal{O}} (a_{\mathbf{p}} \nabla_x \mathbf{p} + b_{\mathbf{p}}\, \varsigma \nabla_x^{\perp} \mathbf{p}) \cdot \nabla_x \Xi\, dx = -\int_{\mathcal{O}} (c_{\mathbf{p}} f + d_{\mathbf{p}}\, \varsigma f^{\perp}) \cdot \nabla_x \Xi\, dx. \tag{19}$$

It satisfies Equation (17) *in the distribution sense and the boundary condition* (18) *in* $H^{-1/2}(\partial\mathcal{O})$. *It is the unique solution of* (17)–(18) *lying in* $\boldsymbol{A}(\mathcal{O})$. $\square$

Proof. The functions $a_{\mathbf{p}}$, $b_{\mathbf{p}}$, $c_{\mathbf{p}}$ and $d_{\mathbf{p}}$ are continuous in $\mathcal{O}$ and, on the shore,

$$a_{\mathbf{p}} \approx \frac{h^2}{\delta^2(\tau_a+\tau_b)}, \quad b_{\mathbf{p}} \approx \frac{\kappa^2 h^4}{2\delta^4(\tau_a+\tau_b)}, \quad c_{\mathbf{p}} \approx -\frac{h}{\delta(\tau_a+\tau_b)}, \quad d_{\mathbf{p}} \approx \frac{2\kappa^2 h^2}{\delta^2(\tau_a+\tau_b)^2}. \tag{20}$$

This may be checked as it is done in Lemma 1 of[6] for $\tau_s = 0$ and $\tau_b = \infty$. Since, see (14), $\tau_a = \tau_s|v_{\mathrm{air}}|$, hypothesis (11) shows that $\tau_a + \tau_b$ is bounded from above and from below by two non negative numbers. Therefore, there exist numbers $\overline{c} \geq \underline{c} > 0$ such that

$$\underline{c}h^2 \leq a_{\mathbf{p}} \leq \overline{c}h^2, \quad |b_{\mathbf{p}}| \leq \overline{c}h^2, \quad |c_{\mathbf{p}}| \leq \overline{c}h, \quad |d_{\mathbf{p}}| \leq \overline{c}h.$$

The left-hand side of (19) is bilinear continuous with respect to $\mathbf{p}$ and Ξ in $\boldsymbol{A}(\mathcal{O})$ since

$$\Big|\int_{\mathcal{O}}(a_{\mathbf{p}}\nabla\mathbf{p} + b_{\mathbf{p}}\varsigma\nabla^{\perp}\mathbf{p})\cdot\nabla\Xi\,dx\Big| \leq 2\overline{c}\int_{\mathcal{O}} h^2|\nabla\mathbf{p}|\,|\nabla\Xi|\,dx \leq 2\overline{c}\,|\mathbf{p}\|_{\boldsymbol{A}(\mathcal{O})}\|\Xi\|_{\boldsymbol{A}(\mathcal{O})}.$$

It is coercive since

$$\int_{\mathcal{O}}(a_{\mathbf{p}}\nabla\mathbf{p} + b_{\mathbf{p}}\varsigma\nabla^{\perp}\mathbf{p})\cdot\nabla\mathbf{p}\,dx = \int_{\mathcal{O}} a_{\mathbf{p}}|\nabla\mathbf{p}|^2\,dx \geq \underline{c}\int_{\mathcal{O}}|h\,\nabla\mathbf{p}|^2\,dx.$$

The right-hand side of (19) is linear continuous with respect to Ξ in $\boldsymbol{A}(\mathcal{O})$ since, see (14), $f = (1+\tau_s)|v_{\mathrm{air}}|v_{\mathrm{air}}$ is bounded in $\mathcal{O}$ by (11).

Then, Lax-Milgram theorem provides the existence and the uniqueness of a solution $\mathbf{p} \in \boldsymbol{A}(\mathcal{O})$ to (19). The equivalence of (19) with (17)–(18) may be proved as it is done in Proposition 12 in[6]. $\square$

7 NUMERICAL SIMULATION OF THE 2D MODEL

Equations (17)–(18) are poorly adapted to numerical calculus because the coefficients $a_{\mathbf{p}}, \ldots, d_{\mathbf{p}}$ degenerate on $\partial\mathcal{O}$. More precisely they behave as $h^m(x)$ with $m \geq 1$ as $h(x) \to 0$, see (20). To overcome this difficulty, let us rewrite these equations using a stream function. If $\mathcal{O}$ is simply connected, that is in the lack of island, $\nabla_x\cdot\overline{\mathbf{v}} = 0$ and $\overline{\mathbf{v}}\cdot n_{\partial\mathcal{O}} = 0$, see (6) and (7), ensure the existence of $\mathbf{q}$ such that $\overline{\mathbf{v}} = \nabla_x^{\perp}\mathbf{q}$. Thanks to (16), (17) and (18), it satisfies

$$\nabla_x\cdot((a_{\mathbf{q}}I + b_{\mathbf{q}}\varsigma A)\nabla_x\mathbf{q} + (\varsigma c_{\mathbf{q}}I + d_{\mathbf{q}}A)f) = 0, \quad \mathbf{q}|_{\partial\mathcal{O}} = 0 \tag{21}$$

where

$$a_{\mathbf{q}} = \frac{a_{\mathbf{p}}}{a_{\mathbf{p}}^2 + b_{\mathbf{p}}^2}, \quad b_{\mathbf{q}} = \frac{-b_{\mathbf{p}}}{a_{\mathbf{p}}^2 + b_{\mathbf{p}}^2}, \quad c_{\mathbf{q}} = \frac{-a_{\mathbf{p}}d_{\mathbf{p}} + b_{\mathbf{p}}c_{\mathbf{p}}}{a_{\mathbf{p}}^2 + b_{\mathbf{p}}^2}, \quad d_{\mathbf{q}} = \frac{a_{\mathbf{p}}c_{\mathbf{p}} + b_{\mathbf{p}}d_{\mathbf{p}}}{a_{\mathbf{p}}^2 + b_{\mathbf{p}}^2}.$$

Moreover

$$\nabla_x \mathbf{p} = (\varsigma b_{\mathbf{q}} I - a_{\mathbf{q}} A)\nabla_x \mathbf{q} + (d_{\mathbf{q}} I - c_{\mathbf{q}} \varsigma A) f. \tag{22}$$

Problem (21) is easier to solve numerically than (17)–(18) since the homogeneous Dirichlet boundary condition replaces Neumann condition and since the coefficients $a_{\mathbf{q}}, \ldots, d_{\mathbf{q}}$ go to infinity on $\partial\mathcal{O}$ (they behave as $h^{-m}(x)$). It defines a unique 2D stream function $\mathbf{q} = \mathbf{q}(x)$ and the 3D velocity $(\mathbf{v}, \mathbf{w})(x, z)$ is given in terms of $\nabla_x \mathbf{q}(x)$, $f(x)$, $h(x)$ and z by (13), (15) and (22).

A numerical simulation of the surface velocity in North Pacific is carried out for Munk reference wind (Figure 1). The stream function $\mathbf{q}$ is computed (Figure 2) using FEM, a finite element method package of O. Pironneau *et al.*[24], on a mesh generated by EMC2 of F. Hecht. Bathymetry is taken equal to 1 in order to compare our results to previous ones[11,21]. The surface velocity $\mathbf{v}_s$ is obtained setting $z = 0$ in (13); it is drawned (Figure 3) by Visu, a postprocess of MODULEF. Our results are similar to those obtained with Homogeneous wind driven model[11] or with Munk model[21]. In particular the western intensification of current known as Kuroshio current is recovered as well as the three main gyres in Ocean.

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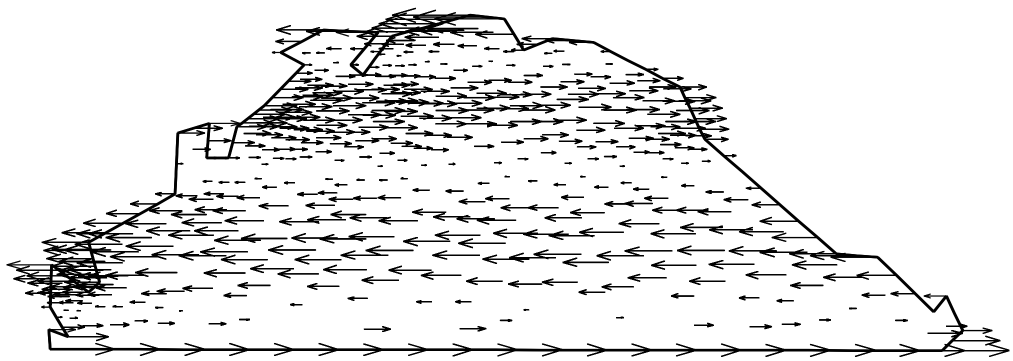

Figure 1: Wind velocity (Munk Profile)

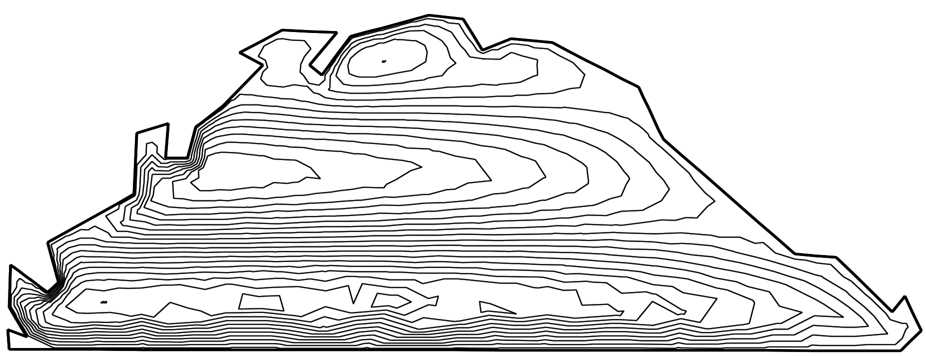

Figure 2: Stream function in North pacific ocean.

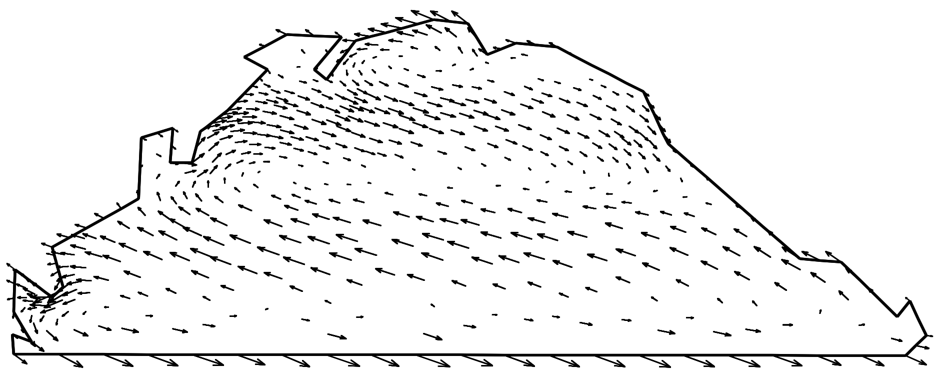

Figure 3: Surface velocity in North pacific ocean.